MAS114 Solutions

Sheet 5 (Week 5)

1. Is it true, for integers a, b, c, that if $a \mid bc$ then either $a \mid b$ or $a \mid c$? Give either a proof or a counterexample.

   Solution No; there are many counterexamples. These include that $4 \mid (2 \times 2)$ but $4 \nmid 2$, and that $6 \mid (10 \times 21)$ but $6 \nmid 10$ and $6 \nmid 21$.

2. Find three numbers a, b and c such that

   - a and b have 20 as a factor (and there's no larger number which is a factor of both),
   - b and c have 6 as a factor (and there's no larger number which is a factor of both),
   - c and a have 14 as a factor (and there's no larger number which is a factor of both).

   Solution You may have found some other example, but the smallest are

   $$\begin{aligned} a &= 2 \times 2 \times 5 \times 7 = 140, \\ b &= 2 \times 2 \times 3 \times 5 = 60, \\ c &= 2 \times 3 \times 7 = 42. \end{aligned}$$

3. Show that all the nine numbers

   $$10! + 2, \quad 10! + 3, \quad \ldots, \quad 10! + 10$$

   are all composite.

Prove similarly that, for any n, there is a block of n consecutive composite integers.

Do you think this is an *efficient* solution to the problem? What is the first block of five consecutive composite numbers? Are the numbers smaller than your approach above gives you?

Solution Since $10! = 2 \times 3 \times \cdots \times 10$, we have

$$2 \mid 10!, \quad 3 \mid 10!, \quad \ldots, \quad 10 \mid 10!.$$

Using Question 1(ii), this means that $2 \mid 10! + 2$, and $3 \mid 10! + 3$, and so on up to $10 \mid 10! + 10$. This means that these numbers are composite.

Similarly, consider the numbers $(n+1)! + k$, for $2 \leq k \leq n+1$. These have the property that $k \mid (n+1)!$, and hence $k \mid (n+1)! + k$, so all these numbers are composite. There are n of them, so we're done.

4. (i) The numbers $3, 5, 7$ are all prime, and are consecutive odd numbers. Does it ever happen again that the three consecutive odd numbers

$$2n+1, 2n+3, 2n+5$$

are all prime?

(ii) Can you find five primes in *arithmetic progression*, that is, numbers a, $a+d$, $a+2d$, $a+3d$, $a+4d$ which are all prime? Can you find six in arithmetic progression?

Given an arithmetic progression $a, \ldots, a+5d$ of six primes, can you say anything about what d must be like? How about for an arithmetic progression of n primes?

Can you find some other configurations of primes that you guess *will* occur repeatedly? (Can you find more of them?) Can you find others that you guess *won't*?

Solution

(i) No, it doesn't. Every odd number is either of the form $6k+1$, $6k+3$ or $6k+5$. If you have three consecutive odd numbers, then you get one of each, but numbers of the form $6k+3$ are multiples of 3 and hence not prime (except for 3 itself).

(ii) The first arithmetic progression of five primes is 5, 11, 17, 23, 29. An arithmetic progression of six primes is 7, 37, 67, 97, 127, 157.

5. Two numbers are said to be *coprime* if their greatest common divisor is 1: in other words, if they have no positive factors in common except 1.

   (i) What's the largest subset of $\{1, \ldots, 20\}$ you can find, in which *every* pair of two different integers is coprime?

   (ii) What's the largest subset of $\{1, \ldots, 20\}$ you can find, in which *no* pair of two different integers is coprime?

   Can you think of any way of proving that these are the largest possible subsets, in each case?

   Solution

   (i) The largest possible such subset has 9 elements: it's $\{1, 2, 3, 5, 7, 11, 13, 17, 19\}$. This is largest because, with the exception of 1 every element must have a prime factor, and that prime factor cannot appear in any of the other elements. There are eight primes less than 20, so we can take them together with 1.

   (ii) The largest possible such subset has 10 elements: it's $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$. This is largest because any set with eleven or more elements must have two consecutives, and consecutive integers are always coprime (by Euclid's algorithm).