MAS114 Solutions

Sheet 7 (Week 8)

1. I went to a shop where widgets cost 36p and gadgets cost 50p, and spent £16.78 on widgets and gadgets. How many of each did I come away with?

   Solution The greatest common divisor of 36 and 50 is 2, and working through Euclid's algorithm gives us

   $$7 \times 36 - 5 \times 50 = 2.$$

   Multiplying through by 839 gives us

   $$(7 \times 839) \times 36 - (5 \times 839) \times 50 = 1678.$$

   Unfortunately, it is not possible to buy -5×839 gadgets, so we need a solution where both numbers are positive.

   The general solution works out to be

   $$(5873 - 25k) \times 36 + (18k - 4195) \times 50 = 1678.$$

   For which k are these all positive? Well, the first bracket means we have to have $k \leq 5873/25$ so $k \leq 234$, and the second bracket means we have to have $k \geq 4195/18$ so $k \geq 234$.

   Hence we bought $5873 - 25 \times 234 = 23$ widgets and $18 \times 234 - 4195 = 17$ gadgets.

2. Make multiplication tables for multiplication modulo 7, 8 and 9.

   In other words, make a 7×7 table whose rows and columns are labelled by the integers $0 \leq n < 7$, and where the entry in row m and column n is the remainder on dividing mn by 7.

Then, similarly, produce an 8×8 table showing remainders on dividing mn by 8, and a 9×9 table showing remainders on dividing by 9.

What properties do they have in common? What properties do only some have, but not others? Can you make guesses about which moduli each property will hold in? (Can you prove any such guesses?) Be as imaginative as you like in answering this!

Solution Well, here they are:

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$\times$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

$\times$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1	3	5	7
3	0	3	6	0	3	6	0	3	6
4	0	4	8	3	7	2	6	1	5
5	0	5	1	6	2	7	3	8	4
6	0	6	3	0	6	3	0	6	3
7	0	7	5	3	1	8	6	4	2
8	0	8	7	6	5	4	3	2	1

3. (i) What is the prime factorisation (ie. what you get by writing it as a product of primes) of the number

$$\gcd(2^3 3^4 5^1 7^3, 2^4 3^2 5^3 7^3)?$$

(ii) What is the prime factorisation of the lowest common multiple of these two numbers?

(iii) Given two integers i and j, explain why we have

$$\min(i,j) + \max(i,j) = i + j,$$

where $\min(i, j)$ is the smaller of the two and $\max(i, j)$ is the larger of the two.

(iv) Given two positive integers a and b, explain why we have

$$\gcd(a, b) \operatorname{lcm}(a, b) = ab.$$

Solution

(i) It's $2^3 3^2 5^1 7^3$.

(ii) It's $2^4 3^4 5^3 7^3$.

(iii) Of the two numbers $\min(i, j)$ and $\max(i, j)$, one will be equal to i and one will be equal to j; hence the sum is the same.

(iv) Let $p_1, p_2, \ldots$ be the primes. The greatest common divisor of the product $p_1^{a_1} p_2^{a_2} \cdots$ and the product $p_1^{b_1} p_2^{b_2} \cdots$ is

$$p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \cdots$$

and the lowest common multiple is

$$p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots .$$

As a result their product is

$$\begin{aligned} p_1^{\min(a_1,b_1)+\max(a_1,b_1)} p_2^{\min(a_1,b_1)+\max(a_2,b_2)} \cdots &= p_1^{a_1+b_1} p_2^{a_2+b_2} \cdots \\ &= (p_1^{a_1} p_2^{a_2} \cdots)(p_1^{b_1} p_2^{b_2} \cdots). \end{aligned}$$

4. There is a far-off country, Onemodtensia, which is much like ours except that the only positive integers known are those of the form $10k + 1$, namely $1, 11, 21, 31, \ldots$. These are the *Onemodtensian positive integers*. They say that one of their numbers $p > 1$ is a *Onemodtensian prime* if it has no Onemodtensian positive integer factors except 1 and p.

(i) Make a list of all Onemodtensian primes between 1 and 491 (by sieving or suchlike).

(ii) Find a counterexample to unique prime factorisation in Onemodtensia: find two different pairs of Onemodtensian primes that multiply to give the same number.

What goes wrong if the Onemodtensian mathematicians try to prove that every number can be written uniquely as a product of primes in the way we did in lectures?

Solution

(i) The only composite numbers in that range are $121 = 11 \times 11, 231 = 21 \times 11, 341 = 31 \times 11, 441 = 21 \times 21, 451 = 11 \times 41$. Hence the others are all prime.

(ii) We have (for example) an equation of Onemodtensian primes

$$51 \times 91 = 21 \times 221.$$

One reasonable point is that the proof of unique factorisation depended on Euclid's algorithm, which can't be implemented in Onemodtensia because it involves subtraction, or something similar. That's true enough, but one could wonder if there was a replacement.

But, in fact, there is not really any such thing as a greatest common divisor in general! Indeed, consider two numbers such as $3 \times 7 \times 13 \times 17$ and $3 \times 7 \times 17 \times 23$. In our familiar natural numbers, their gcd is $3 \times 7 \times 17$, but this is not a Onemodtensian positive integer. In fact there are three Onemodtensian common divisors, 1, 3×7 and 3×17, but neither of the latter two is a divisor of the other.