MAS114: Exercises

December 11, 2020

Note that the challenge problems are intended to be difficult! Doing any of them is an achievement. Please hand them in on a separate piece of paper if you attempt them.

Sets, functions, logic

1. Learn the Greek alphabet: learn the names of all the lower-case letters

$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi, o, \pi, \rho, \sigma, \tau, \upsilon, \phi, \chi, \psi, \omega.$$

   (They're among the commonest of the unfamiliar symbols that mathematicians use. If you're Greek or Cypriot, you have an advantage, but you should still get used to the way their names are pronounced in English.)

2. Which of the following rules define a function? For those that are functions, are they injective? Are they surjective? Are they bijective? Give brief explanations where necessary.

   (i) $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \sqrt{n}$;

   (ii) $g : \mathbb{Z} \to \mathbb{R}$ defined by $g(n) = \sqrt{n}$;

   (iii) $h : \mathbb{Z} \to \mathbb{N}$ defined by $h(n) = |n|$;

   (iv) $i : \mathbb{N} \to \mathbb{N}$ defined by taking $i(n) = 100 - n$.

   (v) $j : \mathbb{Z} \to \mathbb{Z}$ defined by $j(n) = -n$;

   (vi) $k : \mathbb{R} \to \mathbb{Z}$ defined by taking $k(x)$ to be the closest integer to x.

3. (i) Write down an injective function from $\{1, \dots, 10\}$ to $\{1, \dots, 100\}$.

   (ii) Write down a surjective function from $\{1, \dots, 100\}$ to $\{1, \dots, 10\}$.

   (iii) Is there an injection from $\{1, \dots, 100\}$ to $\{1, \dots, 10\}$, or a surjection from $\{1, \dots, 10\}$ to $\{1, \dots, 100\}$?

4. **(Challenge)** How many subsets are there of $\{1, 2, 3, \dots, 19, 20\}$ which contain no two consecutive elements? (For example, $\{1, 4, 18\}$ is okay, but $\{1, 4, 17, 18\}$ is not okay since it contains the consecutives 17 and 18.)

5. Write down all the elements of each of the following sets:

(a) $\{a \in \mathbb{N} \mid a^2 < 9\}$;

(b) $\{a \in \mathbb{N} \mid a^2 \leq 9\}$;

(c) $\{a \in \mathbb{Z} \mid a^2 < 9\}$;

(d) $\{a \in \mathbb{Z} \mid a^2 \leq 9\}$.

6. Consider the following sets:

- $\{1, 2, 4\}$,
- $\{2, 3, 5\}$,
- $\{1, 2, 3, 4, 5\}$,
- $\{2\}$,
- $\{3, 4\}$.

1. Choose three different sets X, Y and Z from the above such that $X \cup Y = Z$.
2. Choose three different sets X, Y and Z from the above such that $X \cap Y = Z$.
3. Choose two different sets X, Y from the above such that $X \cap Y = \emptyset$.

7. Suppose U is a set. If X is a subset of U, we'll write $\overline{X}$ for $U \backslash X$ for the duration of this question. *(This is common notation whenever we work at length with subsets of some particular set.)*

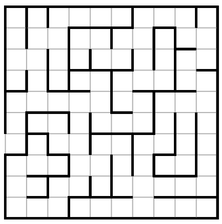
Let A and B be subsets of U. Show that:

(i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(These are called De Morgan's laws. Remember, from lectures: the best way to prove two sets are equal is often to prove that each is contained in the other. So for each of these two you have two containments to prove: the left-hand side in the right-hand side, and vice versa.)

8. **(Challenge)** I foolishly left my dog in a maze, and I'm not allowed in to retrieve him. The maze is a 10 m × 10 m grid, where some of the grid edges are walls and some aren't. I can't remember anything about where the walls are, but this picture shows an example of a similar maze.



My dog is very obedient, but not very clever. He understands the instructions "walk 1 m forwards", "turn 90° left", and "turn 90° right" but nothing else of any use. If I shout for him to walk forwards and that would result in him walking into a wall, he'll just do nothing for that order and wait patiently for the next one, wagging his tail.

I can't see into the maze, and have no idea whether my orders for him to walk forward are succeeding or not.

Is there a sequence of orders I could shout that would get him out of the maze, no matter what the maze looks like and no matter which square I left him at, and no matter which direction I left him facing?

9. An *even number* is an integer that can be written in the form $2k$ for some integer k. An *odd number* is one that can be written in the form $2k+1$ for some integer k. Using these definitions *and no other facts you may happen to know about odd or even numbers*, prove the following implications for integers m and n:

1. If n is even, then n^2 is even.
2. If n and m are odd, then $n + m$ is even.
3. If n and m are odd, then nm is odd.

State the converse of each of the above implications. Do you think they are true or false?

10. Which of these statements is true?

 1. $\forall x \in \mathbb{R}, (x^2 - 7x + 10 = 0) \Rightarrow (x = 2 \wedge x = 5)$
 2. $\forall x \in \mathbb{R}, (x^2 - 7x + 10 = 0) \Rightarrow (x = 2 \vee x = 5)$

11. Can you translate the following statements into English? Which are true and which are false?

 1. $\forall a, b, c \in \mathbb{R}, \exists x \in \mathbb{R} \quad \text{s.t.} \quad ax^2 + bx + c = 0.$
 2. $\forall a, b, c \in \mathbb{C}, \exists x \in \mathbb{C} \quad \text{s.t.} \quad ax^2 + bx + c = 0.$
 3. $\forall a, b, c \in \mathbb{R}, \exists x \in \mathbb{R} \quad \text{s.t.} \quad b^2 - 4ac \geq 0 \Rightarrow ax^2 + bx + c = 0.$

Induction

12. Prove by induction that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

for all positive integers n.

13. **(Challenge)** Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be? You should prove your answer is best.

14. Prove that for all natural numbers $n \geq 2$, we have

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

15. Prove carefully that $4^n < n!$ for all natural numbers $n \geq 10$.

 (The quantity $n!$ denotes the factorial *of n: the product of the natural numbers from 1 up to n.)*

16. The sequence $a_0, a_1, \ldots$ is defined by the following rules:

 - $a_0 = 0$,
 - $a_{2n+1} = 9a_n + 2$ for $n \geq 0$.
 - $a_{2n} = 4a_n + 1$ for $n \geq 1$

 Calculate the first few values $a_1, \ldots, a_5$ by hand. Prove by strong induction that $a_n \geq n^2$ for all n.

17. **(Challenge)** The *Fibonacci numbers* are a function $F : \mathbb{N} \to \mathbb{N}$ defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. So, for example, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, and so on.

 Is $F(2013)$ even or odd? Find an odd prime factor of $F(2013)$.

Elementary number theory

18. For each of the following statements, either prove them or find a counterexample. Your proofs should proceed *directly from the definition of divisibility.*

    (i) Let a, b, c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

    (ii) Let a, b, c be integers. If $a \mid b$ and $a \mid c$, then $a \mid b + c$.

    (iii) Let a, b, c be integers. If $a \mid b + c$, then $a \mid b$ and $a \mid c$.

    (iv) Let a, b, c be integers. If $a \mid b$ and $a \mid c$, then $a \mid bc$.

    (v) Let a, b, c be integers. If $a \mid bc$, then $a \mid b$ and $a \mid c$.

    (vi) Let a, b, c, d be integers. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

19. 1. Show that the product $n(n+1)(n+2)(n+3)$ of any four consecutive numbers is a multiple of 24.

    2. Show that, given any four numbers a, b, c, d whatsoever, the product of their differences

    $$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

    is a multiple of 12.

20. Compute the following, showing your working:

    (a) $\gcd(896, 1200)$;

    (b) $\gcd(123456789, 987654321)$.

21. **(Challenge)** Show that any two numbers of the form $2^{2^n} + 1$ are coprime (where n is a nonnegative integer), and thus give an alternative proof that there are infinitely many primes.

22. Find all integer solutions to the following linear diophantine equations:

    (a) $10x + 17y = 88$;

    (b) $9x + 15y = 100$.

23. Look again at the proof that if a prime p divides ab, then p divides a or b. Simplify it to obtain a proof that, for any integers n, a, and b, if $n \mid ab$ and $\gcd(n, a) = 1$, then $n \mid b$.

24. Let $F(n)$ be the nth Fibonacci number. What is $\gcd(F(n), F(n+1))$? Prove your answer by induction.

25. **(Challenge)** Prove that there are infinitely many primes of the form $4n - 1$. *(Hint: Try thinking about numbers of the form $4p_1p_2 \dots p_k - 1$.)*

Modular arithmetic

26. Find all solutions to the following congruence equations: in each case, either state that there are no solutions, or give them in the form $x \equiv a \pmod{b}$.

    (a) $6x \equiv 10 \pmod{14}$;

    (b) $6x \equiv 9 \pmod{14}$;

    (c) $5x \equiv 8 \pmod{14}$;

    (d) $7x \equiv 8 \pmod{14}$.

27. Let $a_n = 2^{2^n-1}$.

    1. Show that each term is twice the square of the previous one.
    2. What is the behaviour of the sequence, modulo 7?

28. If $a^{31} \equiv 3 \pmod{47}$, show, by cubing both sides and applying Fermat's Little Theorem, that $a \equiv 27 \pmod{47}$.

29. Find all solutions to the congruence equation

$$143x \equiv 243 \pmod{343}.$$

30. 1. Can you find values of n satisfying the following equations?

       (a) $\varphi(n) = 10$ (there is only one solution);

       (b) $\varphi(n) = 20$ (there are five solutions).

31. **(Challenge)** In 1994, Andrew Wiles, building on work of many other people, proved *Fermat's Last Theorem*, that there are no solutions to the equation

$$a^n + b^n = c^n,$$

where a, b, c and n are positive integers and $n > 2$.

Show that there *are* infinitely many solutions in positive integers to

$$a^{34} + b^{34} = c^{35}.$$

Then show that there are also infinitely many solutions in positive integers to

$$a^{51} + b^{52} = c^{53}.$$

32. Solve the following simultaneous congruence equations:

    (a) $x \equiv 5 \pmod{7}$, $\quad x \equiv 2 \pmod{6}$;

    (b) $x \equiv 5 \pmod{8}$, $\quad x \equiv 2 \pmod{6}$;

    (c) $x \equiv 5 \pmod{9}$, $\quad x \equiv 2 \pmod{6}$;

    (d) $x \equiv 17 \pmod{41}$, $\quad x \equiv 36 \pmod{43}$.

33. Here are some three-variable problems!

    1. Solve the simultaneous congruences

    $$\begin{aligned} n &\equiv 4 \pmod 9 \\ n &\equiv 7 \pmod{10} \\ n &\equiv 3 \pmod{11} \end{aligned}$$

    (Hint: solve the first two, and then combine your solution with the last one).

    2. Find all solutions to the equation

    $$10a + 12b + 15c = 1,$$

    where a, b and c are integers. *(Hint: find one solution, subtract as usual to find others, fix one variable and solve for the other two.)*

34. Show that $17 \mid (3^{32} - 2^{32})$ using Fermat's Little Theorem.

35. **(Challenge)**

    (i) Show that 561 is not prime.

    (ii) Show that, even though 561 is not prime, if $\gcd(a, 561) = 1$, then $a^{560} \equiv 1 \pmod{561}$.

    This shows that one possible converse of Fermat's Little Theorem is not true. Numbers with this property, of being composite but "apparently prime" from the point of view of Fermat's Little Theorem, are called Carmichael numbers*.*

The real numbers

36. Show that $\log_{10}(37)$ and $\sqrt[3]{2}$ are irrational numbers.

37. (a) Show directly from the definition of convergence that the sequence defined by

    $$a_n = \frac{1 - \frac{1}{n}}{1 + \frac{1}{n}}$$

    converges to 1.

    (b) Show directly from the definition of convergence that the sequence defined by

    $$b_n = \frac{(2n+1)(2n-1)}{n^2}$$

    converges to 4.

38. **(Challenge)** Show that $\sqrt{3} + \sqrt{5} + \sqrt{7}$ is irrational.