MAS114 Problems

Sheet 4 (Week 4)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. Well-ordering for $\mathbb{N}$. 2. Writing clear and pleasant proofs. 3. Ideas for proving statements. 4. Divisibility. 5. Prime numbers.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

This is intended to be a proof that, for all $x \in \mathbb{R}$ with $x \neq 0$, we have $\left|\frac{1}{x}\right| = \frac{1}{|x|}$. List all the things you can think of which are wrong with this terrible attempt. Then produce a better one, as a team effort:

$$\begin{aligned} \left|\frac{1}{x}\right| &= \frac{1}{|x|} \\ x \geq 0 \Leftrightarrow \frac{1}{x} &\geq 0 \\ \text{so} \quad \frac{1}{x} &= \frac{1}{x} \\ x < 0 \Leftrightarrow \frac{1}{x} &< 0 \\ \text{so} \quad -\frac{1}{x} &= \frac{1}{-x} \\ &= -\frac{1}{x} \quad \square \end{aligned}$$

2

Do the same for this horrid "proof" that, for functions $f : A \to B$ and $g : B \to C$, if f and g are both injective, then so is their composite $g \circ f$:

$$\begin{aligned} a \neq a' &\Rightarrow f(a) \neq f(a') \\ a \neq a' &\Rightarrow g(a) \neq g(a') \\ (g \circ f)(a) &\Rightarrow f(g(a)) \neq f(g(a')) \\ \text{But } g \text{ injective} &\Rightarrow a \neq a'. \\ & \qquad \text{So } g \circ f \text{ injective as required.} \end{aligned}$$

3

Find reasons why the following statements are true for all integer values of a, b and c:

(i) $2 \mid a(a+1)$;

(ii) $2 \mid a(a+1)(a+2)$;

(iii) $3 \mid a(a+1)(a+2)$;

(iv) $6 \mid a(a+1)(a+2)$;

(v) $3 \mid a^3 - 4a$;

(vi) $2 \mid (a-b)(b-c)(c-a)$.

For discussion: Can you make up any similar statements for yourself?

4

Show that each of the following numbers is composite.

- 7777777777777777
- 124211242112421124211242112421
- 123456789
- 10000600009
- 9991
- 14641
- 123456787654321
- 400000001

Given a bit of ingenuity, none of these require a calculator!