MAS114 Problems

Sheet 5 (Week 5)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. There are infinitely many prime numbers. 2. Sieving to find primes. 3. The greatest common divisor. 4. Coprime numbers. 5. Euclid's algorithm.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

Is it true, for integers a, b, c, that if $a \mid bc$ then either $a \mid b$ or $a \mid c$? Give either a proof or a counterexample.

2

Find three numbers a, b and c such that

- a and b have 20 as a factor (and there's no larger number which is a factor of both),
- b and c have 6 as a factor (and there's no larger number which is a factor of both),
- c and a have 14 as a factor (and there's no larger number which is a factor of both).

3

Show that all the nine numbers

$$10! + 2, \quad 10! + 3, \quad \ldots, \quad 10! + 10$$

are all composite.
Prove similarly that, for any n, there is a block of n consecutive composite integers.

For discussion: Do you think this is an *efficient* solution to the problem? What is the first block of five consecutive composite numbers? Are the numbers smaller than your approach above gives you?

4

(i) The numbers $3, 5, 7$ are all prime, and are consecutive odd numbers. Does it ever happen again that the three consecutive odd numbers

$$2n + 1, 2n + 3, 2n + 5$$

are all prime?

(ii) Can you find five primes in *arithmetic progression*, that is, numbers a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$ which are all prime? Can you find six in arithmetic progression?

For discussion: Given an arithmetic progression $a, \ldots, a + 5d$ of six primes, can you say anything about what d must be like? How about for an arithmetic progression of n primes?
Can you find some other configurations of primes that you guess *will* occur repeatedly? (Can you find more of them?) Can you find others that you guess *won't*?

5

Two numbers are said to be *coprime* if their greatest common divisor is 1: in other words, if they have no positive factors in common except 1.

(i) What's the largest subset of $\{1, \ldots, 20\}$ you can find, in which *every* pair of two different integers is coprime?

(ii) What's the largest subset of $\{1, \ldots, 20\}$ you can find, in which *no* pair of two different integers is coprime?

For discussion: Can you think of any way of proving that these are the largest possible subsets, in each case?