MAS114 Problems

Sheet 7 (Week 8)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. Properties of congruence modulo n. 2. Multiplication tables modulo n. 3. Residue classes. 4. Congruence equations.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

I went to a shop where widgets cost 36p and gadgets cost 50p, and spent £16.78 on widgets and gadgets. How many of each did I come away with?

2

Make multiplication tables for multiplication modulo 7, 8 and 9.
In other words, make a 7×7 table whose rows and columns are labelled by the integers $0 \leq n < 7$, and where the entry in row m and column n is the remainder on dividing mn by 7.
Then, similarly, produce an 8×8 table showing remainders on dividing mn by 8, and a 9×9 table showing remainders on dividing by 9.

For discussion: What properties do they have in common? What properties do only some have, but not others? Can you make guesses about which moduli each property will hold in? (Can you prove any such guesses?) Be as imaginative as you like in answering this!

3

(i) What is the prime factorisation (ie. what you get by writing it as a product of primes) of the number

$$\gcd(2^3 3^4 5^1 7^3, 2^4 3^2 5^3 7^3)?$$

(ii) What is the prime factorisation of the lowest common multiple of these two numbers?

(iii) Given two integers i and j, explain why we have

$$\min(i, j) + \max(i, j) = i + j,$$

where $\min(i, j)$ is the smaller of the two and $\max(i, j)$ is the larger of the two.

(iv) Given two positive integers a and b, explain why we have

$$\gcd(a, b) \operatorname{lcm}(a, b) = ab.$$

4

There is a far-off country, Onemodtensia, which is much like ours except that the only positive integers known are those of the form $10k + 1$, namely $1, 11, 21, 31, \ldots$. These are the *Onemodtensian positive integers*. They say that one of their numbers $p > 1$ is a *Onemodtensian prime* if it has no Onemodtensian positive integer factors except 1 and p.

(i) Make a list of all Onemodtensian primes between 1 and 491 (by sieving or suchlike).

(ii) Find a counterexample to unique prime factorisation in Onemodtensia: find two different pairs of Onemodtensian primes that multiply to give the same number.

For discussion: What goes wrong if the Onemodtensian mathematicians try to prove that every number can be written uniquely as a product of primes in the way we did in lectures?