MAS114 Problems

Sheet 10 (Week 11)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. Convergent sequences. 2. Proving convergence.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

(a) Show that "$\sqrt{3}$ is irrational"; in other words, prove that there is no rational number x such that $x^2 = 3$.

(b) If you tried to prove by a similar argument that "$\sqrt{4}$ is irrational": that there was no rational number x such that $x^2 = 4$, where would the first mistake in the proof be?

(c) Show however that "$\sqrt{6}$ is irrational".

For discussion: For which positive integers n do the methods actually work? That is, for which n can you prove that there is no rational x such that $x^2 = n$?

2

We write $\log(x)$ for the natural logarithm (to base e). Show that the sequence $a_3, a_4, a_5, \ldots$ defined by

$$a_n = \frac{1}{\log \log \log(n)}$$

converges to 0. How far do we have to go before $|a_n| < 0.1$?

3

(i) Suppose that $x = a/b$ and $y = c/d$ are rational numbers. Find a simplified form for $\frac{1}{2}(x+y)$. Hence prove that there is a rational number between any two rational numbers.

(ii) Find an example of two different irrational numbers x and y such that $\frac{1}{2}(x+y)$ is irrational.

(iii) Find an example of two different irrational numbers x and y such that $\frac{1}{2}(x+y)$ is rational.

(iv) Suppose that x and y are two different irrational numbers. Let

$$r = \frac{1}{3}(2x+y)$$
$$s = \frac{1}{3}(x+2y)$$

Solve to express x and y in terms of r and s. Hence show that, given that x and y are irrational, r and s cannot both be rational. Hence prove that there is an irrational number between any two irrational numbers.

(v) Suppose that x and y are two different rational numbers. Show (by contradiction) that

$$x + \frac{\sqrt{2}}{2}(y-x)$$

is irrational, and hence show that there is also an irrational number between any two rational numbers.

For discussion: Do you think that there's a rational number between any two irrational numbers?
How does all this influence your picture of how the rational numbers and the irrational numbers sit together within the reals?