

MAS114: Lecture 1

James Cranch

<http://cranch.staff.shef.ac.uk/mas114/>

2021–2022

Welcome!

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Good ways to refer to me include *James* and *Dr Cranch* (and *he/him*).

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Monday 11am
and
Tuesday 10am

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- ▶ Notes will be placed online on the course webpage before each lecture.

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- ▶ The challenge problem is usually hard: you're not expected to attempt it, but might enjoy doing so. Do it on a separate sheet of paper.

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- ▶ There will (very soon) be a discussion board on Blackboard. Please do use it!

If you email me certain sort of questions, I might ask you to put them on the discussion board for the benefit of others.

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If you do not do this, you *will not* be able to catch up in the run-up to the exams.

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- ▶ Leave it until the time of the exam.

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- ▶ *particularly useful* in applications outside mathematics.

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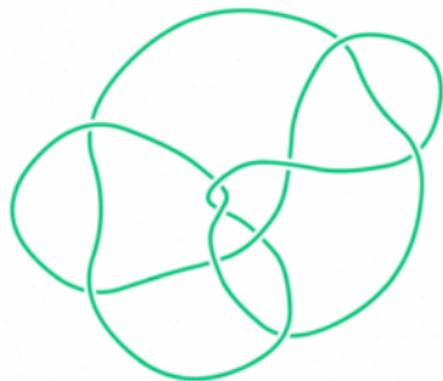
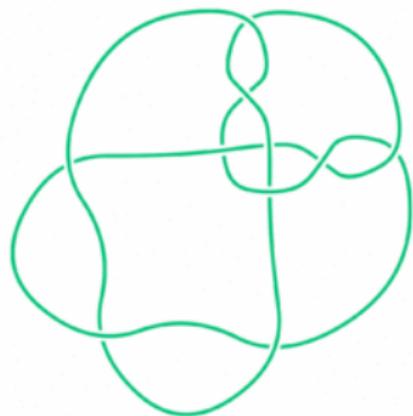
Ideas of space

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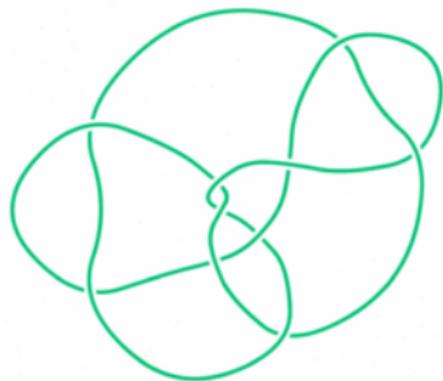
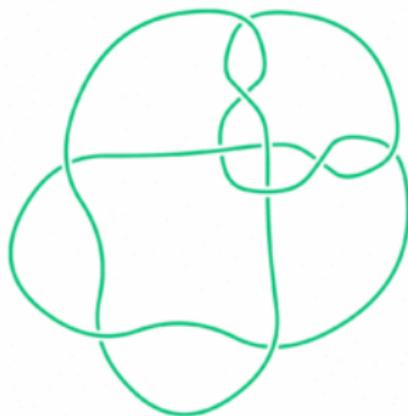
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Are they the same? That is, if I had one, could I manipulate it so as to look like the other?

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Ideas to do with space are nowadays of core importance in physics, just as numbers have. The world is made of space with interesting things in, after all.

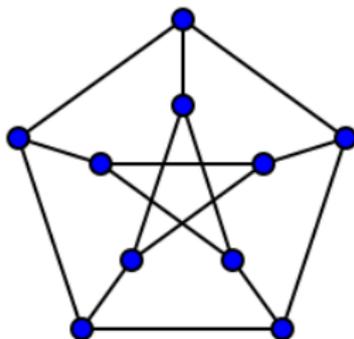
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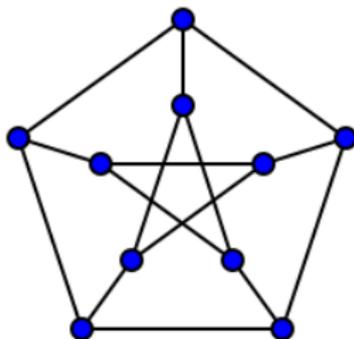
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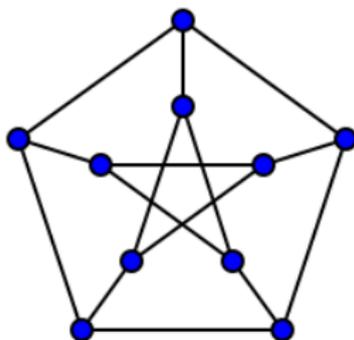
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It's true that the numbers three and ten appear in this problem. But it's not really a problem about numbers: it's a problem about social networks and how they can be configured.

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This has great application in computer science: after all, computer networks are examples of networks.

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Let's look at what that means.

Abstract systems

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When mathematics models the behaviour of a spacerocket, treating the rocket as perfectly round and ignoring the dust and the small lumps of bird mess is the the right way to get an answer that's *good enough*.

One has to be very careful, but the abstraction of mathematics has been an amazing tool. For example, it *may be* true that nothing is perfectly round, but many things are so nearly round as to make their real shape irrelevant.

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If we want to take liberties in our arguments then there's not much point in making an abstraction in the first place.

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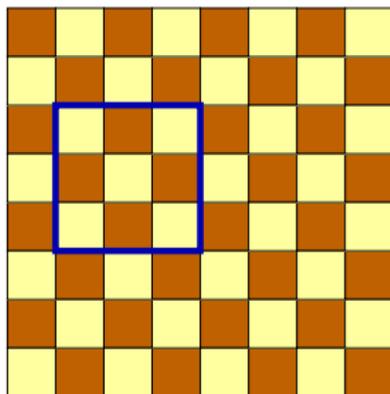
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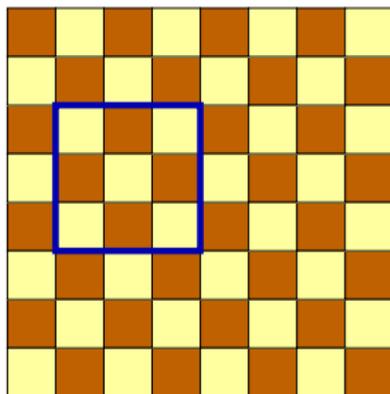
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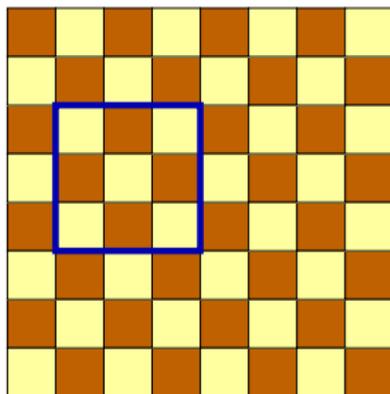


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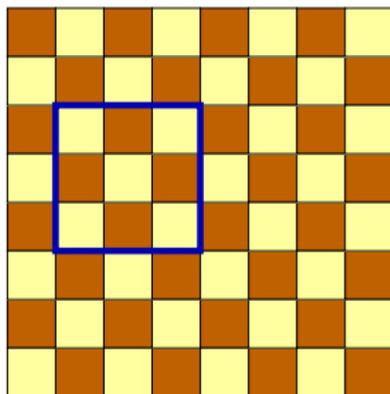


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Here's a vintage map of England and Wales (and, bizarrely, the Isle of Man) coloured in this way:

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- ▶ In **1890**, Percy Heawood pointed out that Kempe's proof contained a big mistake.
- ▶ In **1891**, Julius Petersen pointed out that Tait's proof also contained a big mistake. Now, after twelve years spent believing the problem had been solved, and the answer was yes, mathematicians realised that in fact, they still had no idea.

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- ▶ As of **2021**, the argument of Appel and Haken has been checked many times, and is accepted as a complete solution.

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At times this may seem like an unnecessary burden: especially when you feel that the right answer is “obvious”. However, if you don’t spend time in shallow water learning how to swim, you’ll never be comfortably able to swim in deep water.

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But in order to study them properly, we'll need to start at the beginning, by talking about *sets*.

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We often need to say which we mean, in order to avoid confusion and error. For example, it's certainly possible that I might invite 3 friends over for dinner, but it's hard to invite -5 friends or $3/4$ friends or $\sqrt{2}$ friends over.