MAS114: Lecture 2

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Online tests

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But in order to study them properly, we'll need to start at the beginning, by talking about *sets*.

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We often need to say which we mean, in order to avoid confusion and error. For example, it's certainly possible that I might invite 3 friends over for dinner, but it's hard to invite -5 friends or $3/4$ friends or $\sqrt{2}$ friends over.

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The set of natural numbers is written $\mathbb{N}$ (that's just a letter N, written in a style called "blackboard bold"); in set notation, we might write

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} .$$

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We'll see many more of those curly brackets later!

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Working with a bigger system of numbers can cure this.

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We often use the handy words *non-negative*, meaning "not negative" (in other words, positive or zero) and *non-positive*, meaning "not positive" (in other words, negative or zero).
So the natural numbers are the same thing as the nonnegative integers.

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$-4/9$.

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Of course, any integer n can be regarded as a rational (we can take $\frac{n}{1}$), so

$$\mathbb{Z} \subset \mathbb{Q}.$$

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There are still many things we might want to do but can't do in the rationals though: square roots, logarithms, trigonometry, and suchlike.

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The *real numbers* $\mathbb{R}$ are perhaps the most general sort of numbers you'll have used by now (or perhaps not). They contain lots of the numbers you care about, for example:

$$\pi \in \mathbb{R}, \qquad \log 1729 \in \mathbb{R}, \qquad \sqrt{5} \in \mathbb{R}, \qquad \sin(37^\circ) \in \mathbb{R}.$$

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Producing a good and useful definition of $\mathbb{R}$ is quite tricky, and there wasn't one until about 1870. We'll see one later in the course.

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- $a \in S$ to mean "a is in S".
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- $|S|$ to denote the *size* of S: the number of elements in it. (Of course, some sets are infinite, but this works well for finite ones, at least.)

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Let's write some examples of facts about T using our notation:

$$\textit{Po} \in T, \qquad \textit{Noo-noo} \notin T, \qquad |T| = 4$$

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However, there are few good reasons to write something like that.

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Note that $\varnothing$ is very different to $\{\varnothing\}$. The former, as I mentioned, has no elements; the latter has exactly one element.
That shouldn't confuse you. They're different for pretty much the same reason that "an empty bag" is not the same thing as "a bag which contains an empty bag and nothing else".

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Notice that, for every set A we have

$$A \subset A \qquad \text{and} \qquad \varnothing \subset A.$$