# MAS114: Lecture 5

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2020–2021

# What to call people?

# What to call people?

Since email is the easiest way of getting in touch with me, it's natural to ask what you should call me.

# What to call people?

Since email is the easiest way of getting in touch with me, it's natural to ask what you should call me.
You're welcome to call me James! (Dr Cranch is also fine, but it's rather formal).

# Implication

We were discussing the concept of implication: writing $A \Rightarrow B$ to mean "if $A$, then $B$".

# Converses

# Converses

We've seen in the above, that $A \Rightarrow B$ and $B \Rightarrow A$ are different statements.

# Converses

We've seen in the above, that $A \Rightarrow B$ and $B \Rightarrow A$ are different statements. It's helpful to have a word relating them:

# Converses

We've seen in the above, that $A \Rightarrow B$ and $B \Rightarrow A$ are different statements. It's helpful to have a word relating them:

## Definition

Consider a statement of the form $A \Rightarrow B$. Then the *converse* of that statement is the statement $B \Rightarrow A$.

# Statements and their converses

# Statements and their converses

The truth of an implication has very little to do with the truth of its converse, as we'll see.

# Statements and their converses

The truth of an implication has very little to do with the truth of its converse, as we'll see.

## Example

Let $P$ be the statement "X lives in England" and $Q$ be the statement "X lives in Sheffield".

# Statements and their converses

The truth of an implication has very little to do with the truth of its converse, as we'll see.

### Example

Let $P$ be the statement "X lives in England" and $Q$ be the statement "X lives in Sheffield".
Is the statement $P \Rightarrow Q$ always true?

# Statements and their converses

The truth of an implication has very little to do with the truth of its converse, as we'll see.

## Example

Let $P$ be the statement "X lives in England" and $Q$ be the statement "X lives in Sheffield".
Is the statement $P \Rightarrow Q$ always true?

*No. (X could live in Newcastle.)*

# Statements and their converses

The truth of an implication has very little to do with the truth of its converse, as we'll see.

### Example

Let $P$ be the statement "X lives in England" and $Q$ be the statement "X lives in Sheffield".
Is the statement $P \Rightarrow Q$ always true?

*No. (X could live in Newcastle.)*

Is its converse always true?

# Statements and their converses

The truth of an implication has very little to do with the truth of its converse, as we'll see.

## Example

Let $P$ be the statement "X lives in England" and $Q$ be the statement "X lives in Sheffield".
Is the statement $P \Rightarrow Q$ always true?

*No. (X could live in Newcastle.)*

Is its converse always true?

*Yes, it is.*

# Statements and their converses again

# Statements and their converses again

### Example

Let $C$ be the statement "Y is English", and let $D$ be the statement "Y lives in Sheffield".

# Statements and their converses again

**Example**

Let $C$ be the statement "Y is English", and let $D$ be the statement "Y lives in Sheffield".
Is the statement $C \Rightarrow D$ always true?

# Statements and their converses again

**Example**

Let $C$ be the statement “Y is English”, and let $D$ be the statement “Y lives in Sheffield”.

Is the statement $C \Rightarrow D$ always true?

*No. (Y could live in New York.)*

# Statements and their converses again

**Example**

Let $C$ be the statement “Y is English”, and let $D$ be the statement “Y lives in Sheffield”.

Is the statement $C \Rightarrow D$ always true?

*No. (Y could live in New York.)*

Is the converse always true?

# Statements and their converses again

**Example**

Let $C$ be the statement “Y is English”, and let $D$ be the statement “Y lives in Sheffield”.
Is the statement $C \Rightarrow D$ always true?

*No. (Y could live in New York.)*

Is the converse always true?

*No. (Y could be German.)*

# Equivalence

# Equivalence

Equivalence is the relationship between two statements of being both true, or both false.

# Equivalence

Equivalence is the relationship between two statements of being both true, or both false.

## Definition

Let $P$ and $Q$ be statements. We write $P \Leftrightarrow Q$, pronounced "$P$ is equivalent to $Q$" for the statement that $P$ is true if and only if $Q$ is true.

# Equivalence

Equivalence is the relationship between two statements of being both true, or both false.

## Definition

Let $P$ and $Q$ be statements. We write $P \Leftrightarrow Q$, pronounced "$P$ is equivalent to $Q$" for the statement that $P$ is true if and only if $Q$ is true.

Sometimes people shorten "if and only if" to "iff".

# Negation

# Negation

Negation is a type of opposite:

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced “not $P$”, is the statement “$P$ is false”.

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced "not $P$", is the statement "$P$ is false".

## Remark

We must be careful in thinking of negation as an opposite. For example, the negation of "Richard is happy" is "Richard is not happy".

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced "not $P$", is the statement "$P$ is false".

## Remark

We must be careful in thinking of negation as an opposite. For example, the negation of "Richard is happy" is "Richard is not happy".

Most people would say that the "opposite" is "Richard is sad". That's not quite the same thing!

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced “not $P$”, is the statement “$P$ is false”.

## Remark

We must be careful in thinking of negation as an opposite. For example, the negation of “Richard is happy” is “Richard is not happy”.
Most people would say that the “opposite” is “Richard is sad”.
That’s not quite the same thing!
Similarly, the negation of “Alice is in front of Bob” is not “Alice is behind Bob”, but:

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced "not $P$", is the statement "$P$ is false".

## Remark

We must be careful in thinking of negation as an opposite. For example, the negation of "Richard is happy" is "Richard is not happy".

Most people would say that the "opposite" is "Richard is sad". That's not quite the same thing!

Similarly, the negation of "Alice is in front of Bob" is not "Alice is behind Bob", but:

*"Alice is not in front of Bob".*

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced "not $P$", is the statement "$P$ is false".

## Remark

We must be careful in thinking of negation as an opposite. For example, the negation of "Richard is happy" is "Richard is not happy".

Most people would say that the "opposite" is "Richard is sad". That's not quite the same thing!

Similarly, the negation of "Alice is in front of Bob" is not "Alice is behind Bob", but:

*"Alice is not in front of Bob".*

Note that double negation doesn't do anything: the statement $\neg(\neg P)$ is equivalent to $P$.

# Negation

Negation is a type of opposite:

## Definition

Let $P$ be a statement. The *negation* of $P$, written $\neg P$ and often pronounced "not $P$", is the statement "$P$ is false".

## Remark

We must be careful in thinking of negation as an opposite. For example, the negation of "Richard is happy" is "Richard is not happy".
Most people would say that the "opposite" is "Richard is sad".
That's not quite the same thing!
Similarly, the negation of "Alice is in front of Bob" is not "Alice is behind Bob", but:

*"Alice is not in front of Bob".*

Note that double negation doesn't do anything: the statement $\neg(\neg P)$ is equivalent to $P$. Since statements are either true or false, if it's not "not true", it's true.

# Contrapositives

# Contrapositives

The negation allows us to make sense of something very important about implication:

# Contrapositives

The negation allows us to make sense of something very important about implication:

### Definition

Consider a statement of the form $P \Rightarrow Q$. Its *contrapositive* is the statement $(\neg Q) \Rightarrow (\neg P)$.

# About contrapositives

# About contrapositives

The main useful fact about the contrapositive is that it's equivalent to the original implication. This is something very familiar from everyday life. If my friend Mel says

# About contrapositives

The main useful fact about the contrapositive is that it's equivalent to the original implication. This is something very familiar from everyday life. If my friend Mel says

> *"If I can come visit you this evening, then I'll text you at lunchtime,"*

# About contrapositives

The main useful fact about the contrapositive is that it's equivalent to the original implication. This is something very familiar from everyday life. If my friend Mel says

> *"If I can come visit you this evening, then I'll text you at lunchtime,"*

then I might rephrase it to myself as:

# About contrapositives

The main useful fact about the contrapositive is that it's equivalent to the original implication. This is something very familiar from everyday life. If my friend Mel says

> *"If I can come visit you this evening, then I'll text you at lunchtime,"*

then I might rephrase it to myself as:

> *"I don't get a text at lunchtime, then Mel won't visit this evening."*

# About contrapositives

The main useful fact about the contrapositive is that it's equivalent to the original implication. This is something very familiar from everyday life. If my friend Mel says

> *"If I can come visit you this evening, then I'll text you at lunchtime,"*

then I might rephrase it to myself as:

> *"I don't get a text at lunchtime, then Mel won't visit this evening."*

But here's a formal statement and proof, anyway.

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | | | | |
| 0 | 1 | | | | |
| 1 | 0 | | | | |
| 1 | 1 | | | | |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | | | |
| 0 | 1 | 1 | | | |
| 1 | 0 | 0 | | | |
| 1 | 1 | 1 | | | |

□

# Contrapositives

## Proposition

*Let P and Q be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | | |
| 0 | 1 | 1 | 0 | | |
| 1 | 0 | 0 | 1 | | |
| 1 | 1 | 1 | 0 | | |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | |
| 0 | 1 | 1 | 0 | 1 | |
| 1 | 0 | 0 | 1 | 0 | |
| 1 | 1 | 1 | 0 | 0 | |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | |
| 1 | 0 | 0 | 1 | 0 | |
| 1 | 1 | 1 | 0 | 0 | |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | |
| 1 | 1 | 1 | 0 | 0 | |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 1 |

□

# Contrapositives

## Proposition

*Let $P$ and $Q$ be statements. The statement $P \Rightarrow Q$ is equivalent to its contrapositive $(\neg Q) \Rightarrow (\neg P)$.*

## Proof.

*I'll prove this using a truth table, showing what happens in all possibilities:*

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q$ | $\neg P$ | $(\neg Q) \Rightarrow (\neg P)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 1 |

*You see from this that $(\neg Q) \Rightarrow (\neg P)$ is true exactly when $P \Rightarrow Q$ is, and this proves that they're equivalent.*

□

# “And” and “Or”

# “And” and “Or”

There are other ways to combine statements, other than implication:

# “And” and “Or”

There are other ways to combine statements, other than implication:

**Definition**

Let $P$ and $Q$ be statements. The statement $P \wedge Q$, pronounced “$P$ and $Q$”, is the statement that both $P$ and $Q$ are true.

# “And” and “Or”

There are other ways to combine statements, other than implication:

**Definition**

Let $P$ and $Q$ be statements. The statement $P \wedge Q$, pronounced “$P$ and $Q$”, is the statement that both $P$ and $Q$ are true.
The statement $P \vee Q$, pronounced “$P$ or $Q$”, is the statement that at least one of $P$ or $Q$ (and possibly both) is true.

# Truth tables

# Truth tables

We can make truth tables for both of them:

# Truth tables

We can make truth tables for both of them:

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

# Truth tables

We can make truth tables for both of them:

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

# “Or” in English

# “Or” in English

### Remark

While you’re all used to these words from everyday life, there can be vagueness about how “or” is used in English.

# "Or" in English

### Remark

While you're all used to these words from everyday life, there can be vagueness about how "or" is used in English.

For example,

> *you can have your pie with chips or with mashed potatoes*

is probably intended to mean "but not both".

# “Or” in English

## Remark

While you’re all used to these words from everyday life, there can be vagueness about how “or” is used in English.

For example,

> *you can have your pie with chips or with mashed potatoes*

is probably intended to mean “but not both”. In mathematical argument when we use “or” and mean “but not both”, we have to say so explicitly.

# Implication and “Or”

# Implication and "Or"

It is sometimes worth knowing that implication can be defined in terms of "or":

# Implication and “Or”

It is sometimes worth knowing that implication can be defined in terms of “or”:

## Proposition

*The statement $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$.*

# Implication and "Or"

It is sometimes worth knowing that implication can be defined in terms of "or":

## Proposition

*The statement* $P \Rightarrow Q$ *is equivalent to* $(\neg P) \vee Q$.

## Proof.

The only way that the first statement can be false is if $P$ is true and $Q$ is false. But that's also the only way that the second statement can be false, so they're equivalent. □

# Implication and "Or"

It is sometimes worth knowing that implication can be defined in terms of "or":

## Proposition

*The statement $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$.*

## Proof.

The only way that the first statement can be false is if $P$ is true and $Q$ is false. But that's also the only way that the second statement can be false, so they're equivalent. □

Note that that shows another style of proof of logical statements: by analysis rather than the "case bash" used in truth tables.

# Quantifiers

# Quantifiers

Lastly, we need to discuss how to make statements about *general* situations and *particular* examples.

# Quantifiers

Lastly, we need to discuss how to make statements about *general* situations and *particular* examples. The phrases we use again and again are “for all” and “there exists”: these are called *quantifiers*.

# Quantifiers

Lastly, we need to discuss how to make statements about *general* situations and *particular* examples. The phrases we use again and again are "for all" and "there exists": these are called *quantifiers*. The following symbols are in common use:

# Quantifiers

Lastly, we need to discuss how to make statements about *general* situations and *particular* examples. The phrases we use again and again are “for all” and “there exists”: these are called *quantifiers*. The following symbols are in common use:

$\forall$ for “for all”;

# Quantifiers

Lastly, we need to discuss how to make statements about *general* situations and *particular* examples. The phrases we use again and again are “for all” and “there exists”: these are called *quantifiers*. The following symbols are in common use:

$$\forall \quad \text{for “for all”;}$$
$$\exists \quad \text{for “there exists”;}$$

# Quantifiers

Lastly, we need to discuss how to make statements about *general* situations and *particular* examples. The phrases we use again and again are “for all” and “there exists”: these are called *quantifiers*. The following symbols are in common use:

$$\begin{array}{rc} \forall & \text{for ``for all'';} \\ \exists & \text{for ``there exists'';} \\ \text{s.t.} & \text{for ``such that''.} \end{array}$$

# Examples of quantifiers

# Examples of quantifiers

So, for example,

$$\forall n \in \mathbb{Z}, \quad n^2 - 1 = (n+1)(n-1)$$

# Examples of quantifiers

So, for example,

$$\forall n \in \mathbb{Z}, \quad n^2 - 1 = (n+1)(n-1)$$

is to be read as

*"For all integers n, we have* $n^2 - 1 = (n+1)(n-1)$*."*

# Examples of quantifiers

So, for example,

$$\forall n \in \mathbb{Z}, \quad n^2 - 1 = (n+1)(n-1)$$

is to be read as

*"For all integers n, we have* $n^2 - 1 = (n+1)(n-1)$*."*

And

$$\exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 - 3x - 12 = 0$$

# Examples of quantifiers

So, for example,

$$\forall n \in \mathbb{Z}, \quad n^2 - 1 = (n+1)(n-1)$$

is to be read as

*"For all integers n, we have $n^2 - 1 = (n+1)(n-1)$."*

And

$$\exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 - 3x - 12 = 0$$

is to be read as

*"There exists a real number x such that $x^2 - 3x - 12 = 0$."*

# The importance of quantifiers

# The importance of quantifiers

It’s important that you get used to this notation.

# The importance of quantifiers

It's important that you get used to this notation. This is not because there is anything amazing about it, but because mathematics involves lots of general rules and particular examples: much of the mathematics you do for the next few years will require you to be able to deal with these things.

# The importance of quantifiers

It's important that you get used to this notation. This is not because there is anything amazing about it, but because mathematics involves lots of general rules and particular examples: much of the mathematics you do for the next few years will require you to be able to deal with these things.

One thing you'll have to get used to is situations with two or three quantifiers.

# The importance of quantifiers

It's important that you get used to this notation. This is not because there is anything amazing about it, but because mathematics involves lots of general rules and particular examples: much of the mathematics you do for the next few years will require you to be able to deal with these things.

One thing you'll have to get used to is situations with two or three quantifiers. These happen very frequently: "in general, there is always a particular example of such-and-such"

# The importance of quantifiers

It's important that you get used to this notation. This is not because there is anything amazing about it, but because mathematics involves lots of general rules and particular examples: much of the mathematics you do for the next few years will require you to be able to deal with these things.

One thing you'll have to get used to is situations with two or three quantifiers. These happen very frequently: "in general, there is always a particular example of such-and-such", or "there is a particular amazing example which has the general property of such-and-such".

# The importance of quantifiers

It's important that you get used to this notation. This is not because there is anything amazing about it, but because mathematics involves lots of general rules and particular examples: much of the mathematics you do for the next few years will require you to be able to deal with these things.

One thing you'll have to get used to is situations with two or three quantifiers. These happen very frequently: "in general, there is always a particular example of such-and-such", or "there is a particular amazing example which has the general property of such-and-such".

For example, the statement

$$\forall n \in \mathbb{N}, \quad \exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 = n$$

says that every natural number $n$ has a square root $x$.

# The order of quantifiers

# The order of quantifiers

### Remark

The order of quantifiers is very important. If we swap over the two quantifiers in the last example, we get

$$\exists x \in \mathbb{R} \quad \text{s.t.} \quad \forall n \in \mathbb{N}, \quad x^2 = n.$$

# The order of quantifiers

### Remark

The order of quantifiers is very important. If we swap over the two quantifiers in the last example, we get

$$\exists x \in \mathbb{R} \quad \text{s.t.} \quad \forall n \in \mathbb{N}, \quad x^2 = n.$$

This says that there's a particular number $x$ which has the property that $x$ is the square root of *every natural number*.

# The order of quantifiers

## Remark

The order of quantifiers is very important. If we swap over the two quantifiers in the last example, we get

$$\exists x \in \mathbb{R} \quad \text{s.t.} \quad \forall n \in \mathbb{N}, \quad x^2 = n.$$

This says that there's a particular number $x$ which has the property that $x$ is the square root of *every natural number*. And that's nonsense.

# Negation and quantifiers

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.
The negation of “all Teletubbies are red”

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.
The negation of “all Teletubbies are red”is “not all Teletubbies are red”

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.
The negation of “all Teletubbies are red”is “not all Teletubbies are red”, which is equivalent to “there exists a Teletubby which is not red”.

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.

The negation of “all Teletubbies are red”is “not all Teletubbies are red”, which is equivalent to “there exists a Teletubby which is not red”.

Similarly, the negation of “there exists a dolphin who likes Beethoven”

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.

The negation of “all Teletubbies are red”is “not all Teletubbies are red”, which is equivalent to “there exists a Teletubby which is not red”.

Similarly, the negation of “there exists a dolphin who likes Beethoven”is “there does not exist a dolphin who likes Beethoven”,

# Negation and quantifiers

Another thing that mathematicians have to do every day is understanding how negation interacts with quantifiers.

The negation of “all Teletubbies are red”is “not all Teletubbies are red”, which is equivalent to “there exists a Teletubby which is not red”.

Similarly, the negation of “there exists a dolphin who likes Beethoven”is “there does not exist a dolphin who likes Beethoven”,and that’s equivalent to “all dolphins do not like Beethoven”.

# Negation and Quantifiers II

# Negation and Quantifiers II

In symbols,

# Negation and Quantifiers II

In symbols,

$$\neg\left(\forall x \in X, \quad P(x)\right) \qquad \text{is equivalent to} \qquad \exists x \in X \quad \text{s.t.} \quad \neg P(x).$$

# Negation and Quantifiers II

In symbols,

$$\neg\,(\forall x \in X, \quad P(x)) \qquad \text{is equivalent to} \qquad \exists x \in X \quad \text{s.t.} \quad \neg P(x).$$

$$\neg\,(\exists x \in X \quad \text{s.t.} \quad P(x)) \qquad \text{is equivalent to} \qquad \forall x \in X, \quad \neg P(x).$$

# Negation and Quantifiers II

In symbols,

$$\neg\,(\forall x \in X, \quad P(x)) \qquad \text{is equivalent to} \qquad \exists x \in X \quad \text{s.t.} \quad \neg P(x).$$

$$\neg\,(\exists x \in X \quad \text{s.t.} \quad P(x)) \qquad \text{is equivalent to} \qquad \forall x \in X, \quad \neg P(x).$$

Perhaps you may want to remember that "negation swaps $\forall$ and $\exists$."

# Negation and Quantifiers II

In symbols,

$$\neg\,(\forall x \in X, \quad P(x)) \qquad \text{is equivalent to} \qquad \exists x \in X \quad \text{s.t.} \quad \neg P(x).$$
$$\neg\,(\exists x \in X \quad \text{s.t.} \quad P(x)) \qquad \text{is equivalent to} \qquad \forall x \in X, \quad \neg P(x).$$

Perhaps you may want to remember that “negation swaps $\forall$ and $\exists$.” But being able to *do it correctly by remembering what’s going on* is much more important than remembering a slogan. After a while it should come to seem natural.

# Proofs and counterexamples

# Proofs and counterexamples

Suppose I am wondering whether all fish are slippery.

# Proofs and counterexamples

Suppose I am wondering whether all fish are slippery. If it's true, I need to find some general reason why *every single* fish is slippery. If it's false, I only need to find *one single* fish which isn't slippery, and then I've proved it.

# Proofs and counterexamples

Suppose I am wondering whether all fish are slippery. If it's true, I need to find some general reason why *every single* fish is slippery. If it's false, I only need to find *one single* fish which isn't slippery, and then I've proved it.

In general, if you have a general statement and you don't know if whether it's true or false, then it could either be:

# Proofs and counterexamples

Suppose I am wondering whether all fish are slippery. If it's true, I need to find some general reason why *every single* fish is slippery. If it's false, I only need to find *one single* fish which isn't slippery, and then I've proved it.

In general, if you have a general statement and you don't know if whether it's true or false, then it could either be:

- *true*, in which case you need to prove it in general (that's a statement with a "$\forall$" in);

# Proofs and counterexamples

Suppose I am wondering whether all fish are slippery. If it's true, I need to find some general reason why *every single* fish is slippery. If it's false, I only need to find *one single* fish which isn't slippery, and then I've proved it.

In general, if you have a general statement and you don't know if whether it's true or false, then it could either be:

- *true*, in which case you need to prove it in general (that's a statement with a "$\forall$" in);
- *false*, in which case you need to find a counterexample (that's a statement with a "$\exists$" in).