# MAS114: Lecture 6

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2020–2021

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It's been decided that, to pass the module, you *must* pass both of these components.

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> *If I can reach the bottom (rung number zero?) of a ladder,*
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then the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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### Remark

We call part (i) the *base case*, and part (ii) the *induction step*.
These words agree quite well with our mental picture of a ladder!
When we are trying to prove the induction step $P(k) \Rightarrow P(k+1)$ we refer to $P(k)$ as the *induction hypothesis*.

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We'll prove many things by induction in this course, but this is one:

## Proposition

*For any natural number n, we have the following formula for the sum of the first n positive integers:*

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

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So $P(3)$ is the statement that says $1 + 2 + 3 = 3 \times 4/2$, and $P(10)$ is the statement that

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Notice that $P(n)$ is *not a number*, it's a *statement*.

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For our base case, $P(0)$ says that the sum of *no integers at all* is $0 \times 1/2$, which is true, as the sum of no integers is zero.

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Now we will do our induction step, proving $P(k) \Rightarrow P(k+1)$ for all $k$. Suppose $P(k)$ is true: we need to show that $P(k+1)$ is true.

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$$1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

We need to prove $P(k+1)$, which would say that

$$1 + 2 + \cdots + (k-1) + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

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Now note that

$$\begin{aligned} & 1 + 2 + \cdots + (k-1) + k + (k+1) \\ = \quad & (1 + 2 + \cdots + (k-1) + k) + (k+1) \\ = \quad & \frac{k(k+1)}{2} + (k+1) \quad \text{(by the induction hypothesis)} \\ = \quad & \frac{k(k+1) + 2(k+1)}{2} \\ = \quad & \frac{(k+1)(k+2)}{2}. \end{aligned}$$

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\end{aligned}$$

This is exactly the statement $P(k+1)$, which is what we needed for the induction step, and that completes the proof. □

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If you believe that induction is a reliable method of proof (and I do, and I hope you do too), then it had better be the case that we're not using induction correctly.

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If you don't have a base case, such as $P(0)$, then it's of no use to prove that $P(k) \Rightarrow P(k + 1)$ for all $k$. It's no use to be able to climb a ladder if the bottom of the ladder is unreachable.

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We'll take $P(1)$ as the base case of the induction. This is the statement "Given any one horse, all of them have the same colour": this is obviously true.

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Now we'll prove the induction step. We will assume that $P(k)$ is true ("given any $k$ horses, all of them have the same colour"): our job is to prove that $P(k+1)$ is true ("given any $(k+1)$ horses, all of them have the same colour").

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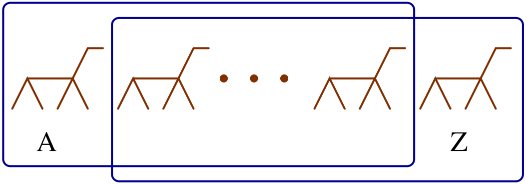

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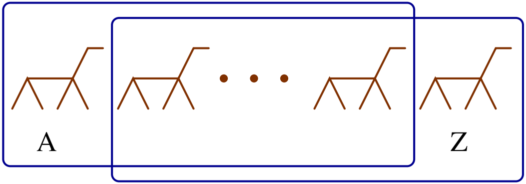


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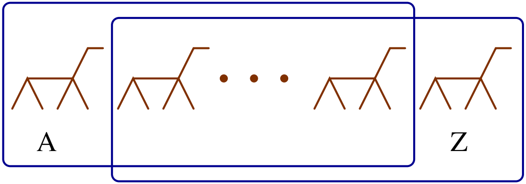


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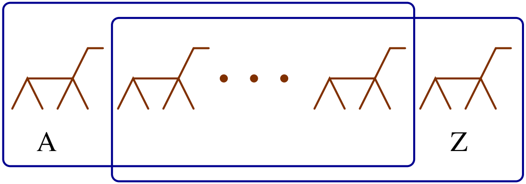


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Also, excluding Zebedee, there are $k$ horses, which all have the same colour, again by the induction hypothesis.

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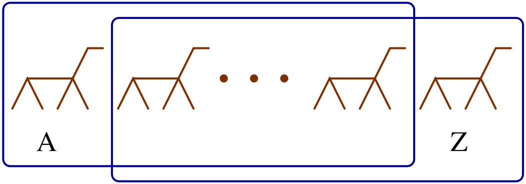


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*The particular case $P(1) \Rightarrow P(2)$ doesn't work.*

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I find this surprisingly subtle.
In fact, it's a parody of a *valid* style of argument. If it is the case that *any two things are the same*, then we could prove using exactly this method that they're *all the same*. In fact, this is something you already know, since "all are alike" and "no two differ" are synonymous phrases.

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(ii) $P(k)$ implies $P(k+1)$ for all $k \geqslant 15$,

then $P(n)$ is true for all $n \geqslant 15$.
Perhaps you want to think of that as saying "if have a door which leads to the fifteenth rung of a ladder, and you know how to climb ladders, then you can get to every rung above the fifteenth".

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Actually, you should have been prepared for this variant: my induction proof that "all horses have the same colour" started with 1, not 0. (Okay, that proof was wrong. But there was nothing wrong with *that bit* of the proof: there's nothing wrong with induction starting from 1. It was something else that was wrong).