MAS114: Lecture 12

James Cranch

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2020–2021

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I'll have an office hour as usual, or will be happy to make an alternative appointment (email me) if you want.

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From week 8 we'll be back to normal.

A recap: Bezout's lemma

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Last time, I proved that, given positive integers a and b, we can write a number e in the form $e = ma + nb$ if and only if $\gcd(a, b)|e$.

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Of course, we should be careful to say what we mean by "only one way". We certainly do have:

$$\begin{aligned} 420 &= 2 \times 2 \times 3 \times 5 \times 7 \\ &= 5 \times 2 \times 3 \times 7 \times 2 \\ &= 7 \times 5 \times 3 \times 2 \times 2, \quad \text{and so on\dots} \end{aligned}$$

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Clearly, what we mean is that every positive integer can be written as a product of primes in only one way, where reordering doesn't count as different. Or, more precisely, that any two ways of writing a positive integer as a product of primes differ only by reordering. Mathematicians say, "in only one way, up to reordering".

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One wants to say something like "as the right-hand side is clearly divisible by 7, the left-hand side must be divisible by 7 too, but there isn't a 7 listed among the primes on the left".
But if we have $7 \mid (487 \times 205339)$, why must we have either $7 \mid 487$ or $7 \mid 205339$? It wouldn't be true if 7 weren't a prime. But this is true for primes!

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Suppose that $p \mid ab$, and consider $\gcd(p, a)$. Since $\gcd(p, a) \mid p$, we either have $\gcd(p, a) = 1$ or $\gcd(p, a) = p$.

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If $\gcd(p, a) = p$, then as $\gcd(p, a) \mid a$, we have $p \mid a$.
If $\gcd(p, a) = 1$, however, then by Bezout's Lemma, we know that there are integers m and n such that $mp + na = 1$. Now suppose we multiply both sides by b; we get $mpb + nab = b$.

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Clearly $p \mid mpb$, and also we have $p \mid nab$ since we are supposing that $p \mid ab$. Hence $p \mid mpb + nab$, so $p \mid b$, as needed. □

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Let p be a prime and let $a_1, \ldots, a_n$ be integers. Then if $p \mid a_1 \cdots a_n$, then $p \mid a_i$ for some i.

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This is an easy induction argument using the result above.

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Suppose not: there is a number n with two genuinely different prime factorisations $n = p_1 \cdots p_r$ and $n = q_1 \cdots q_s$. We can suppose that the p's and the q's have nothing in common (if $p_i = q_j$, then we can cancel them out and use $p_1 \cdots p_{i-1} p_{i+1} \cdots p_r = q_1 \cdots q_{j-1} q_{j+1} \cdots q_s$, which is a smaller example).

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Now, that means that p_1 is different to all of $q_1, q_2, \ldots, q_s$.
We have $p_1 \mid n$, since $n = p_1 \cdots p_r$. But then we also have $p_1 \mid q_1 \cdots q_s$. But by our previous result, this means that $p_1 \mid q_j$ for some j. But, by the definition of q_j being a prime number, that means that $p_1 = q_j$, which we said didn't happen: that gives us our contradiction. □

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The Fermat equation becomes more interesting because of our inability to reliably take nth roots in $\mathbb{Z}$ or $\mathbb{N}$: which x and y can we take for which this recipe works?

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Consider, for example, the equation $39x + 54y = 120$.
This equation would be simple if we cared about real solutions: we could take any x we like and then just take $y = (120 - 39x)/54$.
However, because we can't do division reliably in $\mathbb{Z}$, this recipe is not very helpful: how do we know which x will give us an integer y?
Next lecture, we'll see how to get a general solution.