MAS114: Lecture 17

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We proved that if a and m are coprime, then there is some positive integer n such that $a^n \equiv 1 \pmod{m}$.
The question was, can we calculate such an integer?

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Before we prove it, we'll talk a while longer about invertible elements and multiplication modulo a prime.

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Can we explain this systematically?

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So as there are seven numbers in the list, and one is congruent to each possible residue $0, 1, \ldots, 6$ (modulo 7), they're all different.
This is true in general, for the same reason: if a is coprime to m, then the integers

$$0, a, 2a, \ldots, (m-1)a$$

contain each of the m residues (and so exactly once each, because there's m of them).

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Another is that, since the product contains a copy of every nonzero residue modulo p, it is congruent to $(p-1)!$.

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But all the residues from 1 to $p-1$ are invertible, and the product of invertible residues is invertible, so $(p-1)!$ is invertible. Multiplying both sides by $(p-1)!^{-1}$ leaves us with

$$a^{p-1} \equiv 1 \pmod{p},$$

exactly as promised. □

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For another example, $\varphi(6) = 2$, since 1 and 5 are the only numbers between 1 and 6 which are coprime to 6.

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Theorem (Fermat-Euler Theorem)

Let a and n be integers with $\gcd(a, n) = 1$. Then

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

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Proof.

The proof is exactly the same as Fermat's Little Theorem, but instead of working with all the integers $1, 2, \ldots, n-1$, we just consider those that are invertible modulo n: let's write these as $x_1, x_2, \ldots, x_{\varphi(n)}$.

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If a is invertible, then $ax_1, \ldots, ax_{\varphi(n)}$ are all invertible too, and any invertible residue is of this form: b can be written as $a(a^{-1}b)$.

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Hence if we consider the products of these we have

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Since all the elements $x_1, x_2, \ldots, x_{\varphi(n)}$ are invertible, we can cancel them out to get $a^{\varphi(n)} \equiv 1 \pmod{n}$. □

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Now, this allows us to do this:

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However, if $(n-1)! \equiv -1 \pmod{n}$ and $a|n$, then $(n-1)! \equiv -1 \pmod{a}$, which gives a contradiction.

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consists of one representative of each invertible residue class.

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consists of one representative of each invertible residue class. We can pair each up with its inverse; each element gets paired with another, except for 1 and -1. So, the product consists of a lot of pairs of inverses (whose product modulo n is 1), together with the odd ones out 1 and -1: so the product is -1 as claimed. □

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As a matter of fact, it's not a good way of doing it: if we want to check a large number N, it's quicker to do trial division to see if N has any factors, than it is to multiply lots of numbers together.
But this result was psychologically important in the development of modern fast primality tests: it was the first evidence that there are ways of investigating whether a number N is prime or not by looking at how arithmetic modulo N behaves.