MAS114: Lecture 18

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Alice and Bob are of course named so that the message goes from A to B. Eve is so named because she is an *eavesdropper*, or perhaps because she is *evil*.

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The problem with this old-time approach is that the same secret is used to encrypt and decrypt the message, so needs exchanging.

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5. Alice sends Bob the encrypted message.
6. Bob uses his private key to decrypt it, and read Alice's message.

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Fermat-Euler for pq

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Proposition

Let p and q be different primes. Then the number $\varphi(pq)$, of integers from 1 to pq coprime to pq, is given by

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Hence $q + p - 1$ are not coprime to pq, and so

$$\varphi(pq) = pq - q - p + 1 = (p-1)(q-1).$$

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$$a^{k(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

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Hence, using his private key, Bob can recover what m was from being told m^e.

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Thus his private key is $pq = 10403$, $d = 431$.

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Bob advertises that his public key is $pq = 10403$, $e = 71$. He must work out his private key, by inverting 71 modulo $(p-1)(q-1) = 10200$. A quick use of Euclid's algorithm will do this for him, and he gets that $71^{-1} \equiv 431$. Indeed,

$$71 \times 431 = 30601 = 3 \times 10200 + 1.$$

Thus his private key is $pq = 10403$, $d = 431$.
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Suppose Alice decides she needs to send Bob message 1245, which they've agreed in advance should mean "please meet me after this lecture".

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So she sends Bob 8763.

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So she sends Bob 8763.
Bob receives this, and his task then is to calculate 8763^{431} modulo 10403.

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So she sends Bob 8763.
Bob receives this, and his task then is to calculate 8763^{431} modulo 10403. A similar strategy makes this possible, too, and he finds that

$$8763^{431} \equiv 1245 \pmod{10403},$$

so he has reconstructed Alice's message.

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Let's set ourselves back to a time before $\mathbb{R}$ was invented, and ask: why was it necessary to invent it? Why should we feel that $\mathbb{Q}$ is not enough?
The result that set the ancient Greeks thinking was this:

No rational square root of two

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Theorem

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Proof.

We'll prove this by contradiction; suppose there is such a number $x \in \mathbb{Q}$. Because it's in $\mathbb{Q}$, it takes the form $x = p/q$ for some integers p and q with $q \neq 0$.

We may as well take p and q to be coprime ("in lowest terms").

Then we have $p^2/q^2 = x^2 = 2$, so $p^2 = 2q^2$ with p and q coprime.

Now, the right-hand side is even (it's given as a multiple of 2, so the left-hand side, p^2 must be even too. That means that p itself must be even: so we can write $p = 2r$.

Then we have $(2r)^2 = 2q^2$, which simplifies to $4r^2 = 2q^2$, or $2r^2 = q^2$. Here the left-hand side is even, so q^2 must be even. Hence q itself must be even.

This is a contradiction: p and q can't both be even. So our initial assumption is absurd, and there is no rational x with $x^2 = 2$. $\square$

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The number $\sqrt{2}$ *is not in* $\mathbb{Q}$*.*

But I want to flag that up as being possibly inappropriate: our aim in this section is to define the reals. We shouldn't even be confident that $\sqrt{2}$ exists yet.
However, thanks to this theorem, we can be confident at least that there's no number *inside* $\mathbb{Q}$ which deserves to be called $\sqrt{2}$.

Irrational numbers

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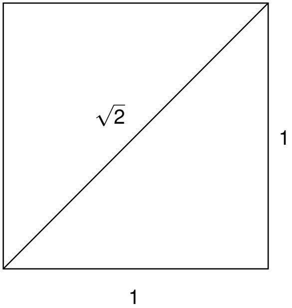

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