MAS114: Lecture 19

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2020–2021

Feedback questionnaires

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Investigating the reals

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For the time being, and *for the time being only* we'll investigate the reals in a similar, informal way. For now, you can regard the real numbers $\mathbb{R}$ as being built out of decimals (as you did at school). In the last lecture of the course, we'll sort this out, and consider a modern construction of the reals.

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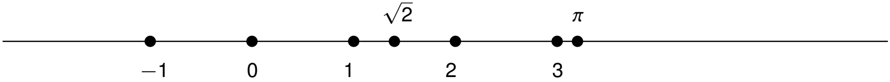

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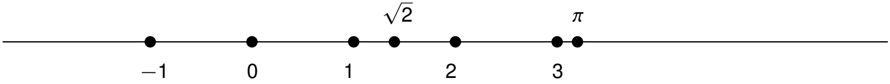

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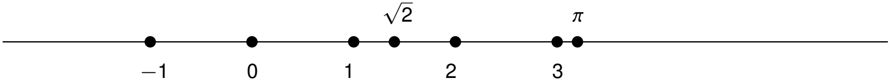

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I've also marked on $\sqrt{2}$, which we now know to be irrational, and π, which I've claimed to you is irrational: these things are in the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers.

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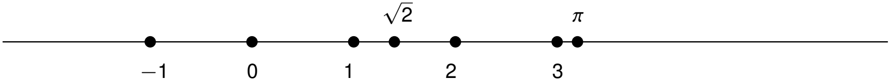

I've marked on the integers -1, 0, 1, 2 and 3, which are all in $\mathbb{Z}$ and hence in $\mathbb{Q}$.
I've also marked on $\sqrt{2}$, which we now know to be irrational, and π, which I've claimed to you is irrational: these things are in the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers.
In my mind, I think of the real numbers $\mathbb{R}$ as a solid line, and the rational numbers $\mathbb{Q}$ as a very fine gauze net stretched out within it.

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The reals $\mathbb{R}$ are also a lovely system of numbers, closed not just those four operations but many others: square roots (of positive numbers), sines, cosines, and so on.

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Can we think of two irrational numbers whose product is rational?

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So, the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ really are just the big messy clump left over in $\mathbb{R}$ when you remove $\mathbb{Q}$.

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We prove the first one by contradiction. Suppose that $x + y$ is rational. Then $(x + y) - y$ is also rational, being obtained by subtracting two rational numbers, but it's equal to x which we know to be irrational. That's the contradiction we wanted.

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Proof.

We prove the first one by contradiction. Suppose that $x + y$ is rational. Then $(x + y) - y$ is also rational, being obtained by subtracting two rational numbers, but it's equal to x which we know to be irrational. That's the contradiction we wanted.
We prove the second one by contradiction too. Suppose that xy is rational. Then $(xy)/y$ is also rational, as it's obtained by dividing two rational numbers (with the latter nonzero), but it's equal to x which we know to be irrational. That's the contradiction we wanted. □

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It turns out that the most interesting things you can ask about are to do with *approximation*. Why is the notion of approximation so important?

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The definition will seem complicated, and probably harder to get your head around than other definitions in the course. However, that's because it really is a subtle concept: all the simpler approaches you might think of are wrong.

Wrong approach 1

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A sequence $a_0, a_1, a_2, \ldots$ *converges to* x if it gets closer and closer to x. In other words, if

$$|a_0 - x| > |a_1 - x| > |a_2 - x| > \cdots .$$

Wrong approach 1, continued

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Why is this completely wrong? Well, for example, the sequence

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also gets closer and closer to 1000:

Wrong approach 1, continued

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Of course, this sequence never gets particularly close to 1000 (the sequence never goes above 4, so it never gets within 996 of 1000), but it's always getting closer!

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But this means that if our definition of “converging to x” were the completely wrong definition “gets closer and closer to x”, then the sequence

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would "converge to π", but it would also "converge to 1000".
But that's not what we want: this sequence is a terrible way of getting to 1000, but a good way of getting to π.

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Before we understand why this is wrong, we ought to make sure we know what this is supposed to mean.
I like to think of it as an argument with a very dangerous and unpleasant *evil opponent*. The evil opponent gets to choose a (positive real) distance, and we win if the sequence gets within that distance of x, and we lose if it doesn't.
In order to be *sure* of winning, we have to know how to beat the evil opponent whatever they say.

Wrong approach 2, continued

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So, in investigating how close the sequence

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gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win.

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On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?",

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On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win.

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On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π,

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On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win.

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On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win. If they ask a question with an even tinier positive number in,

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gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But we can't just hope they'll ask that. Instead they might ask "does the sequence get within distance 0.001 of 1000?", and then we'd lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win. If they ask a question with an even tinier positive number in, we just take more digits and use that and say "yes".

Wrong approach 2, continued

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But according to the definition above it would "converge" to both, because it gets as close as you like to 1 and it also gets as close as you like to 2.

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Definition

Let x be a real number. A sequence of real numbers $a_0, a_1, a_2, \ldots$ is said to *converge to x* if we have

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n > N, \quad |a_n - x| < \epsilon.$$

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So that says "no matter what positive real ϵ our evil opponent gives us, we can point out some N, such that all the terms $a_{N+1}, a_{N+2}, a_{N+3}, \ldots$ are all within ϵ of x".

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That does an excellent job of making precise the concept of "gets close and stays close forever", and it's the right definition!

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Does that converge to 1000? No, it never comes within 1 of 1000 (for example), so it certainly doesn't stay within 1 of 1000 forever.

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What about the sequence

$$a_0 = 1.1, \quad a_1 = 2.01, \quad a_2 = 1.001, \quad a_3 = 2.0001, \quad \ldots ?$$

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No, it doesn't. In particular, it doesn't converge to 1, because while it's sometimes close to 1, it's also sometimes close to 2.

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No, it doesn't. In particular, it doesn't converge to 1, because while it's sometimes close to 1, it's also sometimes close to 2. So there is no N where a_n is always within 0.1 of 1 for all $n > N$: all the odd-numbered a_n aren't in that range.

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No, it doesn't. In particular, it doesn't converge to 1, because while it's sometimes close to 1, it's also sometimes close to 2. So there is no N where a_n is always within 0.1 of 1 for all $n > N$: all the odd-numbered a_n aren't in that range.
Similarly, it doesn't converge to 2, because while it's sometimes close to 2, it's sometimes close to 1. So there is no N where a_n is always within 0.1 of 2 for all $n > N$: all the even-numbered a_n aren't in that range.

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No, it doesn't. In particular, it doesn't converge to 1, because while it's sometimes close to 1, it's also sometimes close to 2. So there is no N where a_n is always within 0.1 of 1 for all $n > N$: all the odd-numbered a_n aren't in that range.
Similarly, it doesn't converge to 2, because while it's sometimes close to 2, it's sometimes close to 1. So there is no N where a_n is always within 0.1 of 2 for all $n > N$: all the even-numbered a_n aren't in that range.
So, given the difficulties we've had in finding the right definition, perhaps you'll have some sympathy for the fact that it took about two centuries to sort real analysis out properly. In what remains of the course I'll try to make you like this definition.