MAS114: Lecture 20

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2020–2021

An early Christmas present

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It's also linked from the main course webpage.

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This embodies the following slogan:

The distance from x to z if we go direct is less than if we go via y.

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Here's a very important fact (which is only true because of all that work we put in finding a good definition):

Proposition

A sequence $a_0, a_1, \ldots$ cannot converge to two different real numbers x and y.

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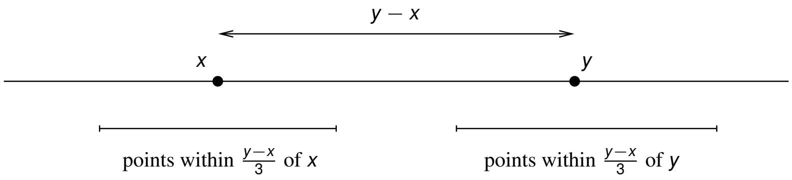

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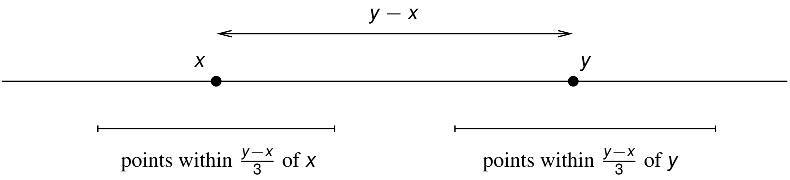

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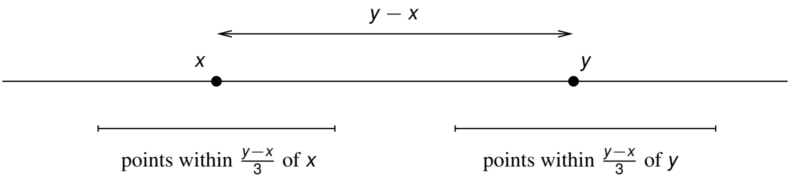

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And since it converges to x and y, eventually all the terms of the sequence should be within the interval around x, and also all of them within the interval around y, which is a contradiction since the intervals don't meet.

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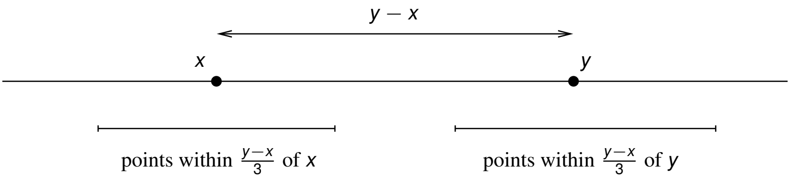

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Let's do the working carefully.

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Proof.

We'll prove this by contradiction. So, suppose it can: suppose that there is a sequence $a_0, a_1, \ldots$, which converges to two different real numbers x and y. Without loss of generality, we may take $x < y$.

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Proof.

We'll prove this by contradiction. So, suppose it can: suppose that there is a sequence $a_0, a_1, \ldots$, which converges to two different real numbers x and y. Without loss of generality, we may take $x < y$. Since the sequence $a_0, a_1, \ldots$ converges to x, there is some N such that, for all $n > N$, we have $|a_n - x| < \frac{y-x}{3}$.

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We'll prove this by contradiction. So, suppose it can: suppose that there is a sequence $a_0, a_1, \ldots$, which converges to two different real numbers x and y. Without loss of generality, we may take $x < y$.

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Since the sequence $a_0, a_1, \ldots$ converges to y, there is some M such that, for all $n > M$, we have $|a_n - y| < \frac{y-x}{3}$.
But then, using the triangle inequality, for any n bigger than both N and M, we have

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$$y - x = |y - x| \leqslant |y - a_n| + |x - a_n| < \frac{y-x}{3} + \frac{y-x}{3} = \frac{2}{3}(y - x)$$

which is a contradiction as $y - x$ is positive. □

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Let's now try proving that some sequence or other does converge, as we're not well practiced at that yet:

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Proposition

The sequence

$$a_1 = 0, \quad a_2 = 1/2, \quad a_3 = 2/3, \quad a_4 = 3/4, \quad a_5 = 4/5, \quad \ldots$$

where $a_n = \frac{n-1}{n}$*, converges to* 1*.*

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Proof (rough version).

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We need to show that, for every $\epsilon > 0$, there is some N, such that for all $n > N$ we have

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A natural thing to do is to simplify that:

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So if we take N to be $\lceil \frac{1}{\epsilon} \rceil$, the smallest integer greater than $1/\epsilon$, that works. □

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exactly as required. □

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That second version is obviously correct, and all the reasoning goes in the right direction. But analysis proofs often have the property that the most persuasive proof will seem a bit mysterious. It's best to do the rough work and then rewrite it neatly.

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That course is about streetfighting, and there you're encouraged to use any techniques you have to hand without worrying too much about what it means.
This is a course about developing fundamental in mathematics and their proofs: if I set problems about convergence in MAS114, I need you to give a rigorous proof, with everything traced back to the definition of convergence (unless you're told otherwise), rather than using the slightly vaguer methods and extra theorems you saw there!