MAS114 Solutions

Sheet 1 (Week 1)

1. Work out the eight terms $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ for each of the following sequences:

   (i) the sequence defined by $a_n = 2^n$ *(this is the "powers of two", or "doubling" sequence)*;

   (ii) the sequence defined by $a_n = (n^3 + 5n + 6)/6$ *(this is called the "cake sequence", as it's the number of pieces you can cut a cake into with n cuts without rearrangement)*;

   (iii) the sequence defined by setting $a_{-3} = a_{-2} = 0$, $a_{-1} = a_0 = 1$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$ for $n \geq 1$;

   (iv) the sequence defined by setting $a_n = (n^4 - 2n^3 + 11n^2 + 14n + 24)/24$;

   (v) the sequence defined by taking a_n to be the sum of the digits of 2^n;

   (vi) the sequence defined by setting a_0 to be 1, and a_n to be 1 plus the sum of all the *individual digits* of all previous terms of the sequence (not the terms themselves, just their digits);

   (vii) the sequence defined by taking a_n to be the remainder left by dividing 2^n by 15.

   This is quite a lot of work, so divide it up between you.

   What does this tell us about the meaning of the phrase "the sequence $1, 2, 4, 8, \ldots$"?

   Solution The sequences begin with the following seven terms:

   (i) $1, 2, 4, 8, 16, 32, 64, 128$;

   (ii) $1, 2, 4, 8, 15, 26, 42, 64$;

(iii) $1, 2, 4, 8, 15, 29, 56, 108$;

(iv) $1, 2, 4, 8, 16, 31, 57, 99$;

(v) $1, 2, 4, 8, 7, 5, 10, 11$;

(vi) $1, 2, 4, 8, 16, 23, 28, 38$;

(vii) $1, 2, 4, 8, 1, 2, 4, 8$.

The moral is that there are *many* sequences beginning $1, 2, 4, 8$, and it may not always be obvious to your reader which you mean.

2. Alice plays a game with Bob. Alice starts by saying either "1", or "$1, 2$", or "$1, 2, 3$". Whatever she says, Bob says either the next one, two, or three natural numbers. Then Alice says the next one, two or three natural numbers, and so on.

   For example, Alice might say "$1, 2$", and then Bob might say "$3, 4, 5$", and then Alice might say "6", and then Bob might say "$7, 8$".

   The *loser* is the player who says "21". If both players play perfectly, which player should win: Alice or Bob?

   Think about what this question even means: what is a winning strategy for a game?

   Solution Note that you win if, on your turn, you manage to reach 20 and then stop after doing so: the next player is then forced to say "21".

   That means that if you manage to reach 16, and then stop, you can win too. If your opponent replies by saying "17", you can say "18, 19, 20" and win; if they reply by saying "17, 18", you can say "19, 20" and win, similarly if they reply "17, 18, 19", you can say "20" and win.

   Similarly, if you manage to reach 12, and then stop, on your next turn you can get to 16 and win by the analysis above. The same holds if you reach 8 (because on your next turn you can reach 12) or 4 (because on your next turn you can reach 8).

   Hence Bob wins, by always replying to Alice in such a way as to finish on 4, 8, 12, 16 or 20.

3. What is the largest number of each of the following types of chess pieces that could be put on an empty 8×8 chessboard so that no piece attacks any other:

(a) rooks?

(b) kings?

(c) bishops?

(d) knights?

(e) queens?

In each of the five cases, try to both find a configuration with as many pieces as possible, and prove that it's not possible to have a larger number on the board. *(If you don't know how chess pieces move, don't just sit about: ask somebody else, or use the internet to find out.)*

Is it convincing to take a board, put a lot of pieces on, and say that no more can be added to that configuration, so it's the maximum?

What would be a completely convincing argument?

Solution

(a) There are eight rows, and each can contain only one rook. So the maximum is eight. This is possible, as shown in the figure.

(b) The board can be divided into 16 squares each measuring 2×2. Each such square can contain at most one king, so the maximum is 16. This is possible, as shown.

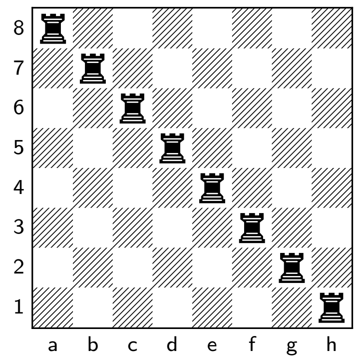
(c) There are 15 diagonals going from southwest to northeast:

a8, a7–b8, a6–c8, a5–d8, a4–e8, a3–f8, a2–g8, a1–h8, b1–h7, c1–h6, d1–h5, e1–h4, f1–h3, g1–h2, h1

Each can contain at most one bishop, and the first and last can contain only one between them. Hence we can't do better than 14, and this is in fact possible, as shown.

(d) One can arrange the whole board into 32 pairs of cells, each a knight's move apart. Pair a1 with c2 and a2 with c1, and a3 with c4 and a4 with c3, and so on down those two rows. Do exactly the same thing with rows b and d, and then with rows e and g, and then f and h. Each of these 32 pairs can contain at most one knight. As a result of this, the maximum possible is 32, and in fact this is possible, as shown.

(e) There are eight rows with at most one queen per row, so the maximum is eight. This is possible, though quite tricky. A solution is shown.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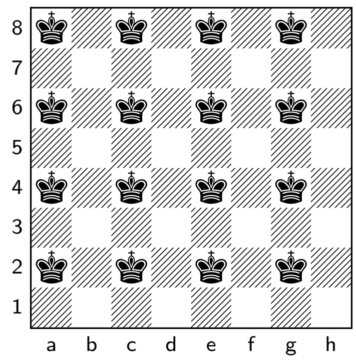
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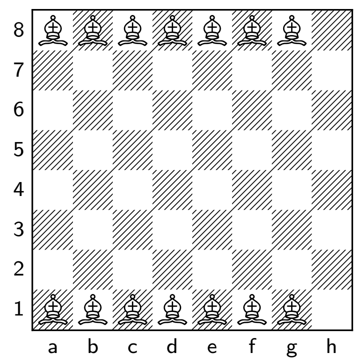
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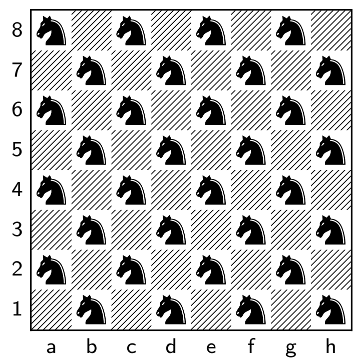
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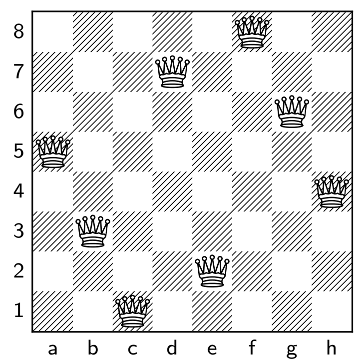
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