MAS114 Solutions

Sheet 2 (Week 2)

1. Find some interesting examples of:

   (i) numbers which are in $\mathbb{Z}$ but not in $\mathbb{N}$;

   (ii) numbers which are in $\mathbb{Q}$ but not in $\mathbb{Z}$;

   (iii) numbers which are in $\mathbb{R}$ but not in $\mathbb{Q}$.

   Which of these sets of numbers have names?

   Solution There are many examples: we just give a few possibilities.

   (i) Examples include -1 and -123456789;

   (ii) Examples include $1/2$ and $-5/7$;

   (iii) Examples include π, $\sqrt{2}$, and $\tan(30°)$.

   $\mathbb{Z} \backslash \mathbb{N}$ is called the "negative numbers", but only if you think that $0 \in \mathbb{N}$.

   $\mathbb{R} \backslash \mathbb{Q}$ is called the "irrational numbers".

2. The *power set* $\mathbb{P}(X)$ of a set X is the set of all subsets of X. For example,

   $$\mathbb{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

   (i) If X is the set $\{A, B, C, D, E\}$, which has size 5, then what is the size of $\mathbb{P}(X)$?

   (ii) Can you find a formula for the size of $\mathbb{P}(Y)$, when Y has size n?

   What is the power set of the empty set? Does this agree with the formula you've come up with in part (ii)?

Solution

(i) Every time we add an element to a set, we double the size of the powerset. Indeed, if A is a set and $x \notin A$, then any subset of A can produce two subsets of $A \cup \{x\}$, depending whether we choose to include or not to include x. This means that starting with $|\mathbb{P}(\emptyset)| = 1$, we double five times to get $|\mathbb{P}(X)| = 32$.

Alternatively, we could just make a tidy table of possibilities:

$\emptyset$	$\{E\}$	$\{D\}$	$\{D, E\}$
$\{C\}$	$\{C, E\}$	$\{C, D\}$	$\{C, D, E\}$
$\{B\}$	$\{B, E\}$	$\{B, D\}$	$\{B, D, E\}$
$\{B, C\}$	$\{B, C, E\}$	$\{B, C, D\}$	$\{B, C, D, E\}$
$\{A\}$	$\{A, E\}$	$\{A, D\}$	$\{A, D, E\}$
$\{A, C\}$	$\{A, C, E\}$	$\{A, C, D\}$	$\{A, C, D, E\}$
$\{A, B\}$	$\{A, B, E\}$	$\{A, B, D\}$	$\{A, B, D, E\}$
$\{A, B, C\}$	$\{A, B, C, E\}$	$\{A, B, C, D\}$	$\{A, B, C, D, E\}$

(ii) The reasoning above shows that the answer is 2^n.

Our formula predicts that the power set of the empty set should have 2^0 elements, or 1. In fact it is $\{\emptyset\}$: the set with just one element, namely the empty set.

3. Let A be a set with m elements, and B a set with n elements.

   How many functions are there from A to B?

   If you're stuck, choose small values of m and n, and experiment. Then make a guess, then try to see why that guess might be true.

   Can you see any link with the previous question?

   Solution There are n^m, because m elements of A can each be sent to n elements of B, independently of each other.

   The link to the previous question is that a subset of A can be represented by a function $A \to \{\text{yes}, \text{no}\}$.

4. The number $\binom{n}{k}$ is defined to be the number of subsets with k-elements of a set that has n elements. So $\binom{4}{2} = 6$, because the set $\{A, B, C, D\}$ has six two-element subsets, namely $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$.

   Find conceptual explanations for the following formulae:

   (i) $\binom{n}{k} = \binom{n}{n-k}$ for all n and k. (So, for example, $\binom{10}{4} = \binom{10}{6}$.)

(ii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

(iii) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ for all n and k (so, for example, $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$.)

(iv) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$.

(v) $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

If you're stuck, choose small values of n, k, etc, and try to find a natural way of matching them up. You're looking to describe sets whose size is given by the numbers on each side, and a reason why they're the same thing.

Solution

(i) Choosing k from n is the same as choosing $n-k$: you're choosing the ones not to choose!

(ii) There are 2^n subsets of a set of size n, and the left-hand side is the zero-element subsets, the one-element subsets, and so on up to the n-element subsets.

(iii) If you have n cats and a dog, then if you're choosing $k+1$ animals, then that's either $k+1$ cats, or the dog and k cats.

(iv) If you have n cats and n dogs, then to choose n animals from it is to choose k cats, and to choose to leave behind k dogs, for some k.

(v) If you have to choose m members out of n for a club, and then choose k of them to be gold members, that's the same as choosing k gold members, and then choosing $m-k$ ordinary members from the remaining $n-k$.