MAS114 Solutions

Sheet 3 (Week 3)

1. Which of the following statements are true?

   (i) $\forall m \in \mathbb{Z}, \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad m < n,$

   (ii) $\forall m \in \mathbb{Z}, \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad m > n,$

   (iii) $\exists m \in \mathbb{Z} \quad \text{s.t.} \quad \forall n \in \mathbb{N}, \quad m < n,$

   (iv) $\exists m \in \mathbb{Z} \quad \text{s.t.} \quad \forall n \in \mathbb{N}, \quad m > n,$

   Can you find a good way of unravelling these statements in your head?

   Solution

   (i) This says that every integer has a natural number bigger than it: that's true.

   (ii) This says that every integer has a natural number smaller than it. This is false: in fact, there is no natural number smaller than -1 (for example).

   (iii) This says there is some integer m such that every natural number is larger than m. That's true: every natural number is larger than -1 (for example).

   (iv) This says there is some integer m such that every natural number is smaller than m. That's false.

2. Let P and Q be statements. Complete the following truth tables, where 1 means true, and 0 means false:

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
0	0	1		0		
0	1	1		1		
1	0	0		1		
1	1	0		1		

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
0	0	1		0		
0	1	1		0		
1	0	0		0		
1	1	0		1		

What is the relationship between the right-hand columns of these two tables? And what does this *mean*? Also, what does this have to do with Question 2 in the homework you've just handed in?

Solution Here they are:

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P) \wedge \neg Q)$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

As to connections with the homework, the other question says that

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B},$$

while this one says that

$$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q) \quad \text{and} \quad \neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q).$$

These are the same, after you replace $\wedge$ with $\cap$, and $\vee$ with $\cup$, and $\neg$ with $\overline{}$).

One explanation is this: suppose we choose an element $x \in X$ and write P to mean the statement "$x \in A$" and Q to mean the statement "$x \in B$".

Then the left-hand equivalence is the natural translation into logic of the equivalence "x is in $\overline{A \cup B}$ if and only if x is in $\overline{A} \cap \overline{B}$". Exactly the same thing holds for the right-hand equivalence.

3. There are a total of sixteen possible truth tables: sixteen different columns of four 0's or 1's. Some of them appeared in the question above, but others didn't.

   Can you find them all, and describe them all using P, Q, $\neg$, $\vee$ and $\wedge$ (and parentheses), in the same way that $P \wedge Q$ describes the column whose values are 0,0,0,1?

   Can you describe some of them in several different ways? Is there any reason to think that some descriptions might be better than others? Is there any other way you'd like to describe them?

Solution Here's a full table, giving some of several possible equivalent descriptions:

0	0	0	0	$P \wedge (\neg P)$
0	0	0	1	$P \wedge Q$
0	0	1	0	$P \wedge (\neg Q)$
0	0	1	1	P
0	1	0	0	$(\neg P) \wedge Q$
0	1	0	1	Q
0	1	1	0	$(P \vee Q) \wedge (\neg(P \wedge Q))$
0	1	1	1	$P \vee Q$
1	0	0	0	$\neg(P \vee Q)$
1	0	0	1	$(P \wedge Q) \vee (\neg(P \vee Q))$
1	0	1	0	$\neg Q$
1	0	1	1	$P \vee (\neg Q)$
1	1	0	0	$\neg P$
1	1	0	1	$(\neg P) \vee Q$
1	1	1	0	$\neg(P \wedge Q)$
1	1	1	1	$P \vee (\neg P)$

These descriptions are mildly annoying: it would be nice to call the bottom columns simply 0 and 1.

4. Continuing the previous question, how many possible truth tables would there be if there were three variables P, Q and R? What if there were four variables, or five?

   Is this a good way of understanding logic, as the number of variables grows large?

   Solution Given n variables, there are 2^n different configurations of each variable being 0 or 1. We must give our truth table a value of 0 or 1 in each case, so that's 2^{2^n} truth tables.

   We have:

$$\begin{aligned} 2^{2^3} &= 64, \\ 2^{2^4} &= 256, \\ 2^{2^5} &= 4294967296, \\ 2^{2^6} &= 18446744073709551616. \end{aligned}$$

   This gets very big very quickly! It would take a huge amount of computing power to go through that last one.

5. Let A be a set with four elements, and B be a set with five elements. How many injections are there from A to B? How many from B to A? How many surjections are there from A to B? How many from B to A?

   Solution There are $5 \times 4 \times 3 \times 2 = 120$ injections from A to B, as we pick a new image for each element in turn.

   There are no injections from B to A, or surjections from A to B.

   To count surjections from B to A, consider that two elements must have the same image. There are 10 ways to choose those two, and then 24 ways to choose what elements that pair and each of the other elements are sent to, meaning there are 240 surjections.