MAS114 Solutions

Sheet 4 (Week 4)

1. List all the things you can think of which are wrong with this (terrible) attempt at a proof. Then produce a better one, as a team effort:

   A "proof" that, for $x \in \mathbb{R}$ with $x \neq 0$, we have $\left|\frac{1}{x}\right| = \frac{1}{|x|}$:

$$\begin{aligned}
\left|\frac{1}{x}\right| &= \frac{1}{|x|} \\
x \geq 0 \Leftrightarrow \frac{1}{x} &\geq 0 \\
\text{so} \quad \frac{1}{x} &= \frac{1}{x} \\
x < 0 \Leftrightarrow \frac{1}{x} &< 0 \\
\text{so} \quad -\frac{1}{x} &= \frac{1}{-x} \\
&= -\frac{1}{x} \quad \square
\end{aligned}$$

Solution I can think of the following major problems:

- nowhere does it tell us what the plan is;
- it doesn't tell us what x may or may not be;
- the proof doesn't actually say anything about absolute values except in a line at the beginning which appears to be a statement of the thing being proved;
- it works backwards, rather than forwards;
- the cases $x \geq 0$ and $x \leq 0$ should be $x > 0$ and $x < 0$, since bad things happen if you try to divide by zero.

2. Do the same for this horrid proof.

   A "proof" that, for functions $f : A \to B$ and $g : B \to C$, if f and g are both injective, then so is their composite $g \circ f$:

$$
\begin{aligned}
a \neq a' &\Rightarrow f(a) \neq f(a') \\
a \neq a' &\Rightarrow g(a) \neq g(a') \\
(g \circ f)(a) &\Rightarrow f(g(a)) \neq f(g(a'))
\end{aligned}
$$

   But g injective $\Rightarrow a \neq a'$.

   So $g \circ f$ injective as required.

   Solution Again, I can think of the following (of course, you may hate it for other reasons):

   - None of the lines are justified (the first two are restatements of the fact that f and g are assumed to be injective; the third is a consequence of the definition of the composite; the fourth appears suddenly);
   - The implication sign is not being used correctly in line four at least: there are other assumptions we're not being told about;
   - When we use the second line, we actually want to use injectivity of g to observe the implication

$$g(f(a)) = g(f(a')) \Rightarrow f(a) = f(a'),$$

     but it's been written in terms of $g(a)$ and $g(a')$ instead;
   - Never is it said what a and a' are.

3. Find reasons why the following statements are true for all integer values of a, b and c:

   (i) $2 \mid a(a+1)$;

   (ii) $2 \mid a(a+1)(a+2)$;

   (iii) $3 \mid a(a+1)(a+2)$;

   (iv) $6 \mid a(a+1)(a+2)$;

   (v) $3 \mid a^3 - 4a$;

   (vi) $2 \mid (a-b)(b-c)(c-a)$.

   Can you make up any similar statements for yourself?

Solution

(a) One of a and $a+1$ must be even (and the other odd) so their product is even.

(b) If $a(a+1)$ is even then so is $a(a+1)(a+2)$.

(c) One of $a(a+1)(a+2)$ is always a multiple of 3, so the product is too. Indeed, if a is of the form $3k$, then a is; if a is of the form $3k+1$, then $a+2$ is, and if a is of the form $3k+2$, then $a+1$ is.

(d) If $2 \mid n$ and $3 \mid n$, then $6 \mid n$, so this follows from the above.

(e) This is similar to those above, since $a^3 - 4a = a(a-2)(a+2)$.

(f) At least two of a, b and c must have the same parity, so their difference is even.

4. Show that each of the following numbers is composite.

- 777777777777777777
- 1242112421124211242112421
- 123456789
- 10000600009
- 9991
- 14641
- 123456787654321
- 400000001

Given a bit of ingenuity, none of these require a calculator!

Solution

- This is a multiple of 7.
- This is a multiple of 12421.
- This is a multiple of 3, as its digits sum to 45.
- This is 100003^2.
- This is $100^2 - 3^3 = 97 \times 103$.
- This is 11^4.
- This is 11111111^2.
- As $4a^4 + b^4 = (2a^2 + 2ab + b^2)(2a^2 - 2ab + b^2)$, this is 19801×20201.