MAS114 Solutions

Sheet 10 (Week 11)

1. (a) Show that "$\sqrt{3}$ is irrational"; in other words, prove that there is no rational number x such that $x^2 = 3$.

   (b) If you tried to prove by a similar argument that "$\sqrt{4}$ is irrational": that there was no rational number x such that $x^2 = 4$, where would the first mistake in the proof be?

   (c) Show however that "$\sqrt{6}$ is irrational".

   For which positive integers n do the methods actually work? That is, for which n can you prove that there is no rational x such that $x^2 = n$?

Solution

(a) We'll prove it by contradiction. Let $\sqrt{3} = p/q$ be rational, with p and q in lowest terms, and thus coprime. Then $p^2/q^2 = 3$, or in other words $p^2 = 3q^2$.

The right-hand side is divisible by 3, and hence the left-hand side is too. That implies that $3 \mid p$, and hence we can write $p = 3r$. So $(3r)^2 = 9r^2 = 3q^2$, which simplifies to $3r^2 = q^2$.

Now, the left-hand side of this is divisible by 3, and hence so is the right-hand side. That means that $3 \mid q$, but that contradicts our assumption that p and q are coprime: we've found a common factor of 3.

(b) Let's see what happens:

> We'll (try to) prove it by contradiction. Let $\sqrt{4} = p/q$ be rational, with p and q in lowest terms, and thus coprime. Then $p^2/q^2 = 4$, or in other words $p^2 = 4q^2$.
>
> The right-hand side is divisible by 4, and hence the left-hand side is too.

Now it all goes wrong: the fact that $4 \mid p^2$ does not imply that $4 \mid p$. In fact, if $p = 2$ we'd have $4 \mid p^2$ but $4 \nmid p$. That means we can't continue the proof.

2. Show that the sequence defined by

$$a_n = \frac{1}{\log\log\log(n)}$$

converges to 0. How far do we have to go before $|a_n| < 0.1$?

Solution We need to show that, for all ϵ, there is some N such that for all $n > N$ we have

$$\left|\frac{1}{\log\log\log(n)}\right| < \epsilon.$$

Rearranging, this says that $\log\log\log n > 1/\epsilon$, or that $n > e^{e^{e^{1/\epsilon}}}$. Hence if we take $N = \lceil e^{e^{e^{1/\epsilon}}} \rceil$, then for $n > N$ we have

$$\left|\frac{1}{\log\log\log(n)}\right| < \frac{1}{\log\log\log(e^{e^{e^{1/\epsilon}}})} = \epsilon.$$

In particular, to get $a_n < 0.1$, we need $N = e^{e^{e^{10}}}$, a very large number.

3. (i) Suppose that $x = a/b$ and $y = c/d$ are rational numbers. Find a simplified form for $\frac{1}{2}(x + y)$. Hence prove that there is a rational number between any two rational numbers.

   (ii) Find an example of two different irrational numbers x and y such that $\frac{1}{2}(x + y)$ is irrational.

   (iii) Find an example of two different irrational numbers x and y such that $\frac{1}{2}(x + y)$ is rational.

   (iv) Suppose that x and y are irrational numbers. Let

$$r = \frac{1}{3}(2x + y)$$
$$s = \frac{1}{3}(x + 2y)$$

   Solve to express x and y in terms of r and s. Hence show that, given that x and y are irrational, r and s cannot both be rational. Hence prove that there is an irrational number between any two irrational numbers.

(v) Suppose that x and y are two different rational numbers. Show (by contradiction) that

$$x + \frac{\sqrt{2}}{2}(y - x)$$

is irrational, and hence show that there is also an irrational number between any two rational numbers.

Do you think that there's a rational number between any two irrational numbers?

How does all this influence your picture of how the rational numbers and the irrational numbers sit together within the reals?

Solution

(i) If $x = a/b$ and $y = c/d$, then

$$\frac{1}{2}(x + y) = \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{ad + bc}{2bd}.$$

This is obviously a rational number.

If $x < y$, we also have

$$x = \frac{1}{2}(2x) < \frac{1}{2}(x + y) < \frac{1}{2}(2y) = y,$$

so this gives us a rational number between any two rational numbers.

(ii) If we take $x = \sqrt{2}$ and $y = 3\sqrt{2}$, then $\frac{1}{2}(x + y) = 2\sqrt{2}$, and all these numbers are irrational.

(iii) If we take $x = -\sqrt{2}$ and $y = \sqrt{2}$, then $\frac{1}{2}(x + y) = 0$.

(iv) We can solve to get $x = 2r - s$ and $y = 2s - r$. That means that, if r and s are rational, then so are x and y. Hence if x and y are irrational, one of r and s must be irrational too.

But if x, y we also have

$$x = \frac{1}{3}(3x) < \frac{1}{3}(2x + y) < \frac{1}{3}(x + 2y) < \frac{1}{3}(3y) = y,$$

so there is an irrational number between any two irrational numbers.

(v) Suppose that x and y are rational, and not equal. If $x + \frac{\sqrt{2}}{2}(y - x)$ is rational, then so is

$$\left(x + \frac{\sqrt{2}}{2}(y - x) \right) - x = \frac{\sqrt{2}}{2}(y - x).$$

But as $y - x$ is a nonzero rational number, that means that

$$\frac{\sqrt{2}}{2}$$

is also rational, and that means that $\sqrt{2}$ is rational. That's a contradiction, so it must be that the quantity we considered originally is irrational.

This number is between x and y, so is what we needed.