MAS114 Solutions

Sheet 11 (Week 12)

1. In this question, we'll write $f(x) \prec g(x)$ to mean "there is an X such that $f(x) < g(x)$ for all $x > X$", which might be thought of informally as saying "when x gets large, eventually we always have $f(x) < g(x)$". So, for example, $x^4 \prec x^5$, as we have $x^4 < x^5$ for all $x > 1$.

   Note that if $f(x) \prec g(x)$, and $g(x) \prec h(x)$, then $f(x) \prec h(x)$.

   (i) For each of the three $\prec$ symbols in the following chain, find an X such that the inequality holds for all $x > X$:

   $$1000000x \ln x \prec 1000x^2 \prec x^{2.001} \prec 1.000001^x.$$

   (ii) Arrange the following functions to form a chain ordered by $\prec$ like the one above:

   - $(\ln x)^{\sqrt{x}}$
   - $(\sqrt{x})^{\ln x}$
   - $\frac{1000x^2}{\ln x}$
   - $\frac{e^{\sqrt{x}}}{1000000}$
   - $\frac{x^2}{\ln(\ln x)}$

   In each of the four places you use it, prove that the relation $\prec$ holds.

   Can you prove that your chain is correctly ordered by $\prec$?

   Given any two functions $f(x)$ and $g(x)$, must we have $f(x) \prec g(x)$, or $g(x) \prec f(x)$, or $f(x) = g(x)$?

Solution

(i) In order to have $1000000x \ln x < 1000x^2$ we need $1000 \ln x < x$, or $\frac{x}{\ln x} > 1000$. By calculus, or suchlike, the left-hand side is increasing, and it is certainly bigger than 1000 for $x > 10000$.

In order to have $1000x^2 < x^{2.001}$ we need $1000 < x^{0.001}$, or $x > 1000^{1000}$.

In order to have $x^{2.001} < 1.000001^x$, by taking logs we get $2.001 \ln x < x \ln 1.000001$, or in other words $\frac{x}{\ln x} > \frac{2.001}{\ln 1.000001} \approx 2001001$. This is certainly true for $x > 10^8$.

(ii) We have

$$\frac{1000x^2}{\ln x} \prec \frac{x^2}{\ln(\ln x)} \prec \left(\sqrt{x}\right)^{\ln x} \prec \frac{e^{\sqrt{x}}}{1000000} \prec (\ln x)^{\sqrt{x}}.$$

2. Note that the statement "the sequence $a_0, a_1, \ldots$ does not converge to x" means "there exists some ϵ, such that for all N there exists $n > N$ such that $|a_n - x| \geq \epsilon$".

   Write down a careful proof, using that, that the sequence defined by $a_n = n$ does not converge to any x.

   Does it feel easier or harder to prove things *do not* converge than that they do? Is this reasonable, or just because you're not used to it?

   Solution Suppose we're trying to prove that the sequence doesn't converge to x. We choose $\epsilon = 1$. Now, for every N, we take $n = 100 + \lceil \max(N, x) \rceil$, and this has the property that

$$|a_n - x| = |100 + \lceil \max(N, x) \rceil - x| \geq 100,$$

   which is what we needed to say that this sequence doesn't converge to x.

3. (i) Prove that $2^n \geq n^2$ for all integers $n \geq 4$.

   (ii) Prove that we have
$$0 \leq \frac{n}{2^n} \leq \frac{1}{n}$$
   for all n.

   (iii) Conclude using the Sandwich Lemma that the sequence $\frac{n}{2^n}$ converges to zero.

   Solution To show that that $n^2 \leq 2^n$ for all n, we use induction.

   This can be proved by induction. For a base case, we have $4^2 \leq 2^4$ (in fact, they're equal).

   For our induction step, we suppose that $k \geq 3$, and $k^2 \leq 2^k$, and try to prove that $(k+1)^2 \leq 2^{k+1}$. But

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&\leq 2k^2 \qquad \text{(since } 2k+1 \leq k^2 \text{ for } k \geq 3\text{)} \\
&\leq 2 \cdot 2^k \qquad \text{(by the induction hypothesis)} \\
&= 2^{k+1}.
\end{aligned}$$

The sequence $(a_i)_{i\in\mathbb{N}}$ obviously converges to 0 (for any ϵ, we can take $N = 0$).

The sequence $(c_i)_{i\in\mathbb{N}}$ converges to 0 by an argument very similar to Proposition 20.92: for any ϵ we can take $N = \lceil 1/\epsilon \rceil$, and then for $n \geq N$ we have

$$|c_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} = \frac{1}{\lceil 1/\epsilon \rceil} \leq \frac{1}{1/\epsilon} = \epsilon.$$

Hence the Sandwich Lemma applies and $(b_i)_{i\in\mathbb{N}}$ converges to 0.

4. Given a sequence $a_1, a_2, a_3, \ldots$, a *subsequence* is something formed by choosing some of the terms only. For example, $a_2, a_3, a_5, a_7, \ldots$ is a subsequence.

   Can you find a sequence $a_1, a_2, a_3, \ldots$ with the property that, for any real number $0 \leq x \leq 1$, there is a subsequence of $a_1, a_2, a_3, \ldots$ which converges to x?

   Solution One of many examples of such a sequence is

$$0.0, 0.1, 0.2, \ldots, 0.9, 0.00, 0.01, \ldots, 0.99, 0.000, \ldots 0.999, 0.0000, \ldots.$$

   For example, the number $\sqrt{2}/2$ is then the limit of the subsequence

$$0.7, 0.70, 0.707, 0.7071, \ldots.$$