MAS114 Problems

Sheet 2 (Week 2)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. Operations on sets. 2. Russell's paradox. 3. The definition of functions. 4. Composition of functions. 5. Injective, surjective, and bijective functions. 6. Logic: implication.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

Find some interesting examples of:

(i) numbers which are in $\mathbb{Z}$ but not in $\mathbb{N}$;

(ii) numbers which are in $\mathbb{Q}$ but not in $\mathbb{Z}$;

(iii) numbers which are in $\mathbb{R}$ but not in $\mathbb{Q}$.

For discussion: Which of these sets of numbers have names?

2

The *power set* $\mathbb{P}(X)$ of a set X is the set of all subsets of X. For example,

$$\mathbb{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\,.$$

(i) If X is the set $\{A, B, C, D, E\}$, which has size 5, then what is the size of $\mathbb{P}(X)$?

(ii) Can you find a formula for the size of $\mathbb{P}(Y)$, when Y has size n?

For discussion: What is the power set of the empty set? Does this agree with the formula you've come up with in part (ii)?

3

Let A be a set with m elements, and B a set with n elements.
How many functions are there from A to B?
If you're stuck, choose small values of m and n, and experiment. Then make a guess, then try to see why that guess might be true.

For discussion: Can you see any link with the previous question?

4

The number $\binom{n}{k}$ is defined to be the number of subsets with k-elements of a set that has n elements. So $\binom{4}{2} = 6$, because the set $\{A, B, C, D\}$ has six two-element subsets, namely $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$.
Find conceptual explanations for the following formulae:

(i) $\binom{n}{k} = \binom{n}{n-k}$ for all n and k. (So, for example, $\binom{10}{4} = \binom{10}{6}$.)

(ii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

(iii) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ for all n and k (so, for example, $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$.)

(iv) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$.

(v) $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

For discussion: If you're stuck, choose small values of n, k, etc, and try to find a natural way of matching them up. You're looking to describe sets whose size is given by the numbers on each side, and a reason why they're the same thing.