

MAS114 Problems

Sheet 4 (Week 4)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. Well-ordering for \mathbb{N} . 2. Writing clear and pleasant proofs. 3. Ideas for proving statements. 4. Divisibility. 5. Prime numbers.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

List all the things you can think of which are wrong with this (terrible) attempt at a proof. Then produce a better one, as a team effort:

A "proof" that, for $x \in \mathbb{R}$ with $x \neq 0$, we have $\left|\frac{1}{x}\right| = \frac{1}{|x|}$:

$$\begin{aligned} \left|\frac{1}{x}\right| &= \frac{1}{|x|} \\ x \geq 0 &\Leftrightarrow \frac{1}{x} \geq 0 \\ \text{so } \frac{1}{x} &= \frac{1}{x} \\ x < 0 &\Leftrightarrow \frac{1}{x} < 0 \\ \text{so } -\frac{1}{x} &= \frac{1}{-x} \\ &= -\frac{1}{x} \quad \square \end{aligned}$$

2

Do the same for this horrid proof.

A “proof” that, for functions $f : A \rightarrow B$ and $g : B \rightarrow C$, if f and g are both injective, then so is their composite $g \circ f$:

$$a \neq a' \Rightarrow f(a) \neq f(a')$$

$$a \neq a' \Rightarrow g(a) \neq g(a')$$

$$(g \circ f)(a) \Rightarrow f(g(a)) \neq f(g(a'))$$

$$\text{But } g \text{ injective} \Rightarrow a \neq a'.$$

So $g \circ f$ injective as required.

3

Find reasons why the following statements are true for all integer values of a , b and c :

(i) $2 \mid a(a + 1)$;

(ii) $2 \mid a(a + 1)(a + 2)$;

(iii) $3 \mid a(a + 1)(a + 2)$;

(iv) $6 \mid a(a + 1)(a + 2)$;

(v) $3 \mid a^3 - 4a$;

(vi) $2 \mid (a - b)(b - c)(c - a)$.

For discussion: Can you make up any similar statements for yourself?

4

Show that each of the following numbers is composite.

- 7777777777777777777
- 1242112421124211242112421
- 123456789
- 10000600009
- 9991
- 14641
- 123456787654321
- 400000001

Given a bit of ingenuity, none of these require a calculator!