MAS114 Problems

Sheet 11 (Week 12)

Preamble

Themes from this week (ask your tutorial staff if you're stuck): 1. The Sandwich Lemma. 2. General facts about convergent sequences. 3. Cauchy sequences. 4. Constructing the real numbers.

Work on these problems one at a time in small groups (of around four). For many problems there is designed to be a lot of work that's best shared; and for others discussion is vital to understanding.

1

In this question, we'll write $f(x) \prec g(x)$ to mean "there exists an X such that, for all $x > X$, we have $f(x) < g(x)$". Informally, we think of this as saying "when x gets large, eventually we always have $f(x) < g(x)$".
So, for example, $2x^4 \prec x^5$, since we have $2x^4 < x^5$ whenever $x > 2$.

(i) Show that the following all hold:

  (a) $1000x^{10} \prec x^{11}$
  (b) $1000x^{10} \prec x^{10.001}$
  (c) $\sqrt{x} \prec \frac{x}{\ln x}$
  (d) $x^{1000} \prec 1.001^x$
  (e) $1000000x \prec e^{\sqrt{x}}$
  (f) $\left(\sqrt{x}\right)^{\ln x} \prec (\ln x)^{\sqrt{x}}$

  Note that, for some of these, you'll be able to work out the exact point where the inequality first becomes true. In others, you won't. That's okay: you don't need to!

(ii) Given any two functions $f(x)$ and $g(x)$, must we have $f(x) \prec g(x)$, or $g(x) \prec f(x)$, or $f(x) = g(x)$? Prove it, or find a counterexample.

(iii) If we have $f(x) \prec g(x)$, and $g(x) \prec h(x)$, then must it be true that $f(x) \prec h(x)$? Prove it, or find a counterexample.

2

Note that the statement "the sequence $a_0, a_1, \ldots$ does not converge to x" means "there exists some ϵ, such that for all N there exists $n > N$ such that $|a_n - x| \geq \epsilon$". Write down a careful proof, using that, that the sequence defined by $a_n = n$ does not converge to any x.

For discussion: Does it feel easier or harder to prove things *do not* converge than that they do? Is this reasonable, or just because you're not used to it?

3

(i) Prove that $2^n \geq n^2$ for all integers $n \geq 4$.

(ii) Prove that we have

$$0 \leq \frac{n}{2^n} \leq \frac{1}{n}$$

for all n.

(iii) Conclude using the Sandwich Lemma that the sequence $\frac{n}{2^n}$ converges to zero.

4

Given a sequence $a_1, a_2, a_3, \ldots$, a *subsequence* is something formed by choosing some of the terms only. For example, $a_2, a_3, a_5, a_7, \ldots$ is a subsequence.
Can you find a sequence $a_1, a_2, a_3, \ldots$ with the property that, for any real number $0 \leq x \leq 1$, there is a subsequence of $a_1, a_2, a_3, \ldots$ which converges to x?