MAS114: Solutions to Exercises

Up to week 2

Note that the challenge problems are intended to be difficult! Doing any of them is an achievement. Please hand them in on a separate piece of paper if you attempt them.

Sets, functions, logic

1. Learn the Greek alphabet: learn the names of all the lower-case letters

$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi, o, \pi, \rho, \sigma, \tau, \upsilon, \phi, \chi, \psi, \omega.$$

(They're among the commonest of the unfamiliar symbols that mathematicians use. If you're Greek or Cypriot, you have an advantage, but you should still get used to the way their names are pronounced in English.)

Solution You're on your own with this one (but millions of primary school children in two countries have done it, so it can't be very difficult).

2. Which of the following rules define a function? For those that are functions, are they injective? Are they surjective? Are they bijective? Give brief explanations where necessary.

(i) $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \sqrt{n}$;
(ii) $g : \mathbb{Z} \to \mathbb{R}$ defined by $g(n) = \sqrt{n}$;
(iii) $h : \mathbb{Z} \to \mathbb{N}$ defined by $h(n) = |n|$;
(iv) $i : \mathbb{N} \to \mathbb{N}$ defined by taking $i(n) = 100 - n$.
(v) $j : \mathbb{Z} \to \mathbb{Z}$ defined by $j(n) = -n$;
(vi) $k : \mathbb{R} \to \mathbb{Z}$ defined by taking $k(x)$ to be the closest integer to x.

Solution

(i) This is a function, as every natural number has a real square root. It's injective, because every natural has a different square root. It's not surjective, for example because $1/2$ is not the square root of a natural number. Hence it's also not bijective.
(ii) This is not a function, because -1 does not have a real square root.

(iii) This is a function: every integer has an absolute value which is a natural number. It's not injective, because $h(-1) = h(1) = 1$. But is surjective, since for every $n \in \mathbb{N}$, we have $h(n) = n$. It can't be bijective, as it isn't injective.

(iv) This is not a function: we have $i(101) = -1 \notin \mathbb{N}$.

(v) This is a function: every integer can be negated to give another integer. It's injective, since no two different integers have equal negations. It's surjective, since for any $n \in \mathbb{Z}$ we have $j(-n) = n$. Hence it's bijective too!

(vi) This is "almost" a function, but in fact it isn't. For example, there is no integer closest to $1/2$, as 0 and 1 are equally close.

3. (i) Write down an injective function from $\{1, \dots, 10\}$ to $\{1, \dots, 100\}$.

   (ii) Write down a surjective function from $\{1, \dots, 100\}$ to $\{1, \dots, 10\}$.

   (iii) Is there an injection from $\{1, \dots, 100\}$ to $\{1, \dots, 10\}$, or a surjection from $\{1, \dots, 10\}$ to $\{1, \dots, 100\}$?

Solution

(i) $f(n) = n$ will do (there are many others)!

(ii) One possibility is to take $g(n) = \lceil n/10 \rceil$ (that is, the smallest integer greater than or equal to $n/10$), so that

$$\begin{aligned} g(1) = g(2) = \cdots = g(10) = 1, \\ g(11) = g(12) = \cdots = g(20) = 2, \\ \dots g(91) = g(92) = \cdots = g(100) = 10. \end{aligned}$$

There are many ways to describe this function, and many other functions which work.

(iii) No. For there to be an injection from $\{1, \dots, 100\}$ to $\{1, \dots, 10\}$, there would have to be 100 different objects in the codomain; for there to be a surjection from $\{1, \dots, 10\}$ to $\{1, \dots, 100\}$ there would have to be at most 10 elements in the codomain.

4. **(Challenge 1)** How many subsets are there of $\{1, 2, 3, \dots, 19, 20\}$ which contain no two consecutive elements? (For example, $\{1, 4, 18\}$ is okay, but $\{1, 4, 17, 18\}$ is not okay since it contains the consecutives 17 and 18.)