MAS114: Easter challenge problems

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Here are a few harder-than-usual problems on the themes of the second semester of MAS114.

If you fancy giving them a go, I'm happy to read attempts at these problems during or after the Easter holiday: you can email them to me, or deliver them to my office (G39c Hicks), or leave them in my pigeonhole in the Hicks I floor common room. If you give me solutions on paper, please make sure your email address is written on them clearly.

1. There are 48 symmetries of the cube. Find the order of all of them.

2. Show that the number of elements of the symmetric group S_{2n} which can be written as a product of exactly n disjoint transpositions is
$$\frac{(2n)!}{2^n(n!)}.$$

3. Find a formula for $|\mathrm{GL}_2(\mathbb{Z}_p)|$ (the order of the group of invertible 2×2 matrices with entries modulo p), where p is a prime number.

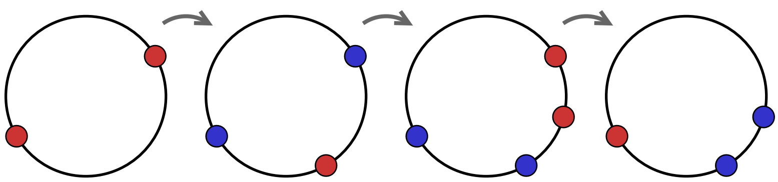
4. I have a necklace, which may have red beads and blue beads on it. A move consists of either:

   (i) adding a red bead at some point, and changing the colour of both of the two neighbouring beads (from red to blue, or from blue to red); or

   (ii) removing a red bead, and changing the colour of both of the two neighbouring beads (again, from red to blue, or from blue to red, as appropriate).

   If I start with two red beads and no blue beads on the necklace, can I use these moves to reach a configuration where I have two blue beads and no red beads?

   (Below is a sequence of three possible starting moves, to illustrate the rules.)



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