MAS114 Homework Solutions

Week 2 (hand in in week 3)

1. Write down all the elements of each of the following sets:

   (a) $\{a \in \mathbb{N} \mid a^2 < 9\}$;

   (b) $\{a \in \mathbb{N} \mid a^2 \leq 9\}$;

   (c) $\{a \in \mathbb{Z} \mid a^2 < 9\}$;

   (d) $\{a \in \mathbb{Z} \mid a^2 \leq 9\}$.

   Solution The sets are:

   (a) $\{0, 1, 2\}$;

   (b) $\{0, 1, 2, 3\}$;

   (c) $\{-2, -1, 0, 1, 2\}$;

   (d) $\{-3, -2, -1, 0, 1, 2, 3\}$.

2. Suppose U is a set. If X is a subset of U, we'll write $\overline{X}$ for $U \backslash X$ for the duration of this question. *(This is common notation whenever we work at length with subsets of some particular set.)*

   Let A and B be subsets of U. Show that:

   (i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

   (ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

   (These are called De Morgan's laws. Remember, the best way to prove two sets are equal is often to prove that each is contained in the other. So for each of these two you have two containments to prove: the left-hand side in the right-hand side, and vice versa.)

Solution

(i) We'll show that each set is contained in the other, showing first that $\overline{A \cup B} \subset \overline{A} \cap \overline{B}$. Indeed, if $x \in \overline{A \cup B}$, then $x \in U$ and $x \notin A \cup B$. But that means that $x \notin A$ and $x \notin B$, or, equivalently, that $x \in \overline{A}$ and $x \in \overline{B}$. Hence $x \in \overline{A} \cap \overline{B}$, as required.

Now we'll show the converse containment, namely that $\overline{A} \cap \overline{B} \subset \overline{A \cup B}$. The argument above is reversible: if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}$ and $x \in \overline{B}$, and hence $x \notin A$ and $x \notin B$. This gives us that $x \notin A \cup B$, or equivalently that $x \in \overline{A \cup B}$.

These two containments show that the two sets are equal.

(ii) As before, we'll show that each set is contained in the other. We'll show first that $\overline{A \cap B} \subset \overline{A} \cup \overline{B}$. But, if $x \in \overline{A \cap B}$, then $x \in U$ and $x \notin A \cap B$. As a result, we have $x \notin A$ or $x \notin B$.

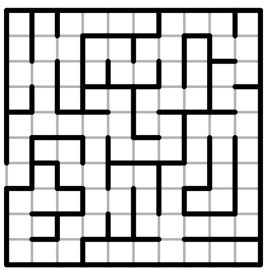
If $x \notin A$, then $x \in \overline{A}$ and hence $x \in \overline{A} \cup \overline{B}$. But if $x \notin B$, then $x \in \overline{B}$ and hence $x \in \overline{A} \cup \overline{B}$ all the same. This proves the first containment.

Now we'll show the other one: that $\overline{A} \cup \overline{B} \subset \overline{A \cap B}$. However, if $x \in \overline{A} \cup \overline{B}$, then $x \in U$, and $x \in \overline{A}$ or $x \in \overline{B}$.

If the former is true, $x \in \overline{A}$, then $x \notin A$ and so $x \notin A \cap B$, and so $x \in \overline{A \cap B}$. If the latter is true, then $x \notin B$ and so $x \notin A \cap B$ and so $x \in \overline{A \cap B}$ all the same.

This completes the proof.

3. **Challenge:** I foolishly left my dog in a maze, and I'm not allowed in to retrieve him. The maze is a 10 m × 10 m grid, where some of the grid edges are walls and some aren't. I can't remember anything about where the walls are, but this picture shows an example of a similar maze.



My dog is very obedient, but not very clever. He understands the instructions "walk 1 m forwards", "turn 90° left", and "turn 90° right" but nothing else of any use. If I shout for him to walk forwards and that would result in him walking into a wall, he'll just do nothing for that order and wait patiently for the next one, wagging his tail.

I can't see into the maze, and have no idea whether he is able to walk forwards or not after each time I tell him to do so.

Is there a sequence of orders I could shout that would get him out of the maze, no matter what the maze looks like and no matter which square I left him at, and no matter which direction I left him facing?

[*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]