MAS114 Homework Solutions

Week 3 (hand in in week 4)

1. An *even number* is an integer that can be written in the form $2k$ for some integer k. An *odd number* is one that can be written in the form $2k+1$ for some integer k. Using these definitions *and no other facts you may happen to know about odd or even numbers*, prove the following implications:

   (a) If n is even, then n^2 is even.

   (b) If n and m are odd, then $n+m$ is even.

   (c) If n and m are odd, then nm is odd.

   State the converse of each of the above implications. Do you think they are true or false?

   Solution For the first part:

   (a) If n is even, then $n = 2k$ for some integer k. That means that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is of the form $2l$. Hence n^2 is even.

   (b) If n and m are odd, then we can write $n = 2k+1$ and $m = 2l+1$ for some integers k and l. That means that $n+m = (2k+1)+(2l+1) = 2k+2l+2 = 2(k+l+1)$. This is of the form $2a$, so $n+m$ is even.

   (c) Again, if n and m are odd, then we can write $n = 2k+1$ and $m = 2l+1$ for some integers k and l. That means that $nm = (2k+1)(2l+1) = 4kl+2k+2l+1 = 2(2kl+k+l)+1$. That means that nm is odd.

   For the converses:

   (a) The converse is "if n^2 is even, then n is even", which is true.

(b) The converse is "if $n + m$ is even, then n and m are odd". That need not be true, since we could have $n = 4$ and $m = 6$ (for example).

(c) The converse is "if nm is odd, then n and m are odd". That's true.

2. Prove by induction that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

for all natural numbers n.

Solution This proof is by induction on n, as requested.

For our base case, we show it's true for $n = 0$. In this case, the left-hand side is the sum of no integers, and is hence zero. The right-hand side is also zero, and so they're equal.

Now we do the induction step: we assume that it's true for $n = k$, and we try deducing that it's true for $n = k + 1$.

In other words, we assume that

$$1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{,}$$

and we try to prove that

$$1^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

But this is not too hard, since we have both

$$\begin{aligned}
&1 + \cdots + k^2 + (k+1)^2 \\
=&\left(1 + \cdots + k^2\right) + (k+1)^2 \\
=&\frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the induction step)} \\
=&\frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
=&\frac{(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)}{6} \\
=&\frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

and also

$$\begin{aligned}
&\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
=&\frac{(k+1)(k+2)(2k+3)}{6} \\
=&\frac{2k^3+9k^2+13k+6}{6}.
\end{aligned}$$

Hence the two are equal, as required. This completes the induction proof.

3. **Challenge:** Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be? You should prove your answer is best.

   [*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]