MAS114 Homework Solutions

Week 4 (hand in in week 5)

1. Prove that for all natural numbers $n \geq 2$, we have

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Solution Let's prove it by induction on n. Our base case is where $n = 2$, where we must prove

$$\left(1-\frac{1}{2^2}\right)=\frac{2+1}{2\times 2},$$

which is true since both sides are equal to $3/4$.

Now, for the induction step, we'll assume it true for $n = k$ and prove it for $n = k+1$. So we're assuming

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k},$$

and must prove

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2(k+1)}.$$

But

$$
\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)\\
=&\left(\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right)\left(1-\frac{1}{(k+1)^2}\right)\\
=&\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)\\
=&\frac{k+1}{2k}\frac{(k+1)^2-1}{(k+1)^2}\\
=&\frac{k+1}{2k}\frac{k(k+2)}{(k+1)^2}\\
=&\frac{k+2}{2(k+1)},
\end{aligned}
$$

as required. Note that, in the above, the second equation is due to the induction hypothesis. This completes the proof.

2. Prove carefully that $4^n < n!$ for all natural numbers $n \geq 10$.

   (The quantity $n!$ denotes the factorial *of n: the product of the natural numbers from 1 up to n.)*

   Solution We'll prove that $4^n < n!$ for all $n \geq 10$ by induction on n, starting with the base case of 10 itself.

   By calculating, we find that $4^{10} = 1048576$, while $10! = 3628800$, so $4^{10} < 10!$.

   Now, for our induction step, suppose that $4^k < k!$ for some $k \geq 10$; we'll show that $4^{k+1} < (k+1)!$. Indeed, we have

   $$4^{k+1} = 4.4^k < (k+1)k! = (k+1)!.$$

   Here the first equation is by definition of exponentiation. The inequality is because $4 < (k+1)$ (as $k \geq 10$ by assumption) and because of the induction hypothesis (that $4^k < k!$). And lastly, the final equation is by the definition of the factorial.

3. **Challenge:** The *Fibonacci numbers* are a function $F : \mathbb{N} \to \mathbb{N}$ defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. So, for example, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, and so on.

Is $F(2013)$ even or odd? Find an odd prime factor of $F(2013)$.

[*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]