MAS114 Homework Solutions

Week 5 (hand in in week 6)

1. For each of the following statements, either prove them or find a counterexample. Your proofs should proceed *directly from the definition of divisibility.*

   (i) Let a, b, c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

   (ii) Let a, b, c be integers. If $a \mid b$ and $a \mid c$, then $a \mid b + c$.

   (iii) Let a, b, c be integers. If $a \mid b + c$, then $a \mid b$ and $a \mid c$.

   (iv) Let a, b, c be integers. If $a \mid b$ and $a \mid c$, then $a \mid bc$.

   (v) Let a, b, c be integers. If $a \mid bc$, then $a \mid b$ and $a \mid c$.

   (vi) Let a, b, c, d be integers. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Solution

(i) This is true: to say that $a \mid b$ is to say that $am = b$ for some integer m. To say that $b \mid c$ is to say that $bn = c$ for some integer c. If both of these are true, then we have $amn = bn = c$, and hence $a \mid c$, as required.

(ii) This is true: if $a \mid b$ and $a \mid c$, we have that $am = b$ for some $m \in \mathbb{Z}$, and $an = c$ for some $n \in \mathbb{Z}$. This means that $a(m + n) = am + an = b + c$, and hence $a \mid b + c$.

(iii) This is false: consider that $2 \mid 2$, but $2 \nmid 1$ and $2 \nmid 1$.

(iv) This is true: since $b \mid bc$, it follows immediately from the first part.

(v) This is false: we have $4 \mid 4$, but $4 \nmid 2$ and $4 \nmid 2$.

(vi) This is true: if $a \mid b$ and $c \mid d$, then $am = b$ for some $m \in \mathbb{Z}$, and $cn = d$ for some $n \in \mathbb{Z}$. This means that $acmn = bd$, and so $ac \mid bd$ as required.

2. **Challenge:** Show that any two numbers of the form $2^{2^n} + 1$ are coprime (where n is a nonnegative integer), and thus give an alternative proof that there are infinitely many primes.

   [*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]