MAS114 Homework Solutions

Week 6 (hand in in week 8)

1. Find all integer solutions to the following linear diophantine equations:

   (a) $10x + 17y = 88$;

   (b) $9x + 15y = 100$.

Solution

(a) We can use Euclid to find that $\gcd(10, 17) = 1$. By working backwards through Euclid's algorithm one can find a solution to $10x + 17y = 1$ and multiply both sides by 88 to get a solution. However, one may prefer to spot that $10 \times 2 + 17 \times 4 = 88$.

Suppose (x, y) is another solution, we have

$$10 \times 2 + 17 \times 4 = 88$$
$$10x + 17y = 88,$$

and so (by subtracting),

$$10(x - 2) + 17(y - 4) = 0.$$

This means that $y - 4$ must be a multiple of 10, so $y - 4 = 10k$. But then $x - 2 = -17k$. Thus

$$x = 2 - 17k, \qquad y = 10k + 4.$$

We've shown that if x and y are solutions, then they're of these form; we should also check that any x and y of this form are solutions. But

$$10(2 - 17k) + 17(10k + 4) = 20 + 68 = 88,$$

so it does work, and so this is the general solution.

(b) The greatest common divisor of 9 and 15 is 3, so the left-hand-side can never equal 100, which is not a multiple of 3. So there are no solutions.

2. Look again at the proof that if a prime p divides ab, then p divides a or b. Simplify it to obtain a proof that, for any integers n, a, and b, if $n \mid ab$ and $\gcd(n, a) = 1$, then $n \mid b$.

   Solution If $\gcd(n, a) = 1$, then there are integers u and v with $nu + av = 1$. Multiplying both sides by b, we get $nub + avb = b$. The integer n divides the left-hand side since it divides n and ab, so we have $n \mid b$.

3. **Challenge:** Prove that there are infinitely many primes of the form $4n - 1$. *(Hint: Try thinking about numbers of the form $4p_1p_2 \ldots p_k - 1$.)*

   [*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]