MAS114 Homework Solutions

Week 8 (hand in in week 9)

1. Find all solutions to the following congruence equations: in each case, either state that there are no solutions, or give them in the form $x \equiv a \pmod{b}$.

   (a) $6x \equiv 10 \pmod{14}$;

   (b) $6x \equiv 9 \pmod{14}$;

   (c) $5x \equiv 8 \pmod{14}$;

   (d) $7x \equiv 8 \pmod{14}$.

Solution

(a) We would like $10 - 6x = 14k$, or in other words, $6x + 14k = 10$. It's easy to check that $\gcd(6, 14) = 2$, and a solution is given by $k = 5$, $x = -10$.

Other solutions occur where $6(x+10) = 14(k-5)$, which is equivalent to $3(x + 10) = 7(k - 5)$. That happens when $3(x + 10)$ is a multiple of 7, which in turn is when $(x + 10)$ is a multiple of 7, which is when $x \equiv 4 \pmod{7}$.

(b) We would like $6x + 14k = 9$. But $\gcd(6, 14) = 2$, and so this equation has no solutions.

(c) We would like $5x + 14k = 8$. We have that $\gcd(5, 14) = 1$, and $-1 \times 14 + 3 \times 5 = 1$, so $k = -8, x = 24$ is a solution.

Other solutions happen when $5(x - 24) = 14(k + 8)$, which boils down to $x - 24$ being a multiple of 14, in other words $x \equiv 10 \pmod{14}$.

(d) We would like $7x + 14k = 8$. But $\gcd(7, 14) = 7$, and so there are no solutions.

2. Find all solutions to the congruence equation

$$143x \equiv 243 \pmod{343}.$$

Solution We seek solutions to

$$143x + 343k = 243.$$

We'll do it by taking the gcd, and working backwards, as usual.

We have

$$\begin{aligned}
\gcd(143, 343) &= \gcd(143, 2 \times 143 + 57) \\
&= \gcd(143, 57) = \gcd(57, 143) \\
&= \gcd(57, 2 \times 57 + 29) \\
&= \gcd(57, 29) = \gcd(29, 57) \\
&= \gcd(29, 1 \times 29 + 28) \\
&= \gcd(29, 28) = \gcd(28, 29) \\
&= \gcd(28, 1 \times 28 + 1) \\
&= \gcd(28, 1) = 1.
\end{aligned}$$

That means that

$$\begin{aligned}
1 &= 29 - 28 \\
&= 29 - (57 - 29) = 2(29) - 57 \\
&= 2(143 - 2(57)) - 57 = 2(143) - 5(57) \\
&= 2(143) - 5(343 - 2(143)) = 12(143) - 5(343).
\end{aligned}$$

So one way of getting $143x \equiv 1 \pmod{343}$ is to take $x = 12$; the general solution for that equation satisfies

$$143(x - 12) + 343(k - 5) = 0.$$

When this happens, we have $343 \mid (x - 12)$, and so $x = 12 + 343n$ for some n.

So the general solution to the equation we were given is to take $x \equiv 12 \cdot 243 \equiv 172 \pmod{343}$.

3. **Challenge:** In 1994, Andrew Wiles, building on work of many other people, proved *Fermat's Last Theorem*, that there are no solutions to the equation

$$a^n + b^n = c^n,$$

where a, b, c and n are positive integers and $n > 2$.

Show that there *are* infinitely many solutions in positive integers to

$$a^{34} + b^{34} = c^{35}.$$

Then show that there are also infinitely many solutions in positive integers to

$$a^{51} + b^{52} = c^{53}.$$

[*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]