MAS114 Homework Solutions

Week 10 (hand in in week 11)

1. Show that $\log_{10}(37)$ and $\sqrt[3]{2}$ are irrational numbers.

   Solution

   (a) We'll prove it by contradiction. Suppose that $\log_{10}(37)$ is rational, and let p and q be integers (with q positive) such that $p/q = \log_{10}(37)$.
   Then $10^{p/q} = 37$, so $10^p = (10^{p/q})^q = 37^q$. But the right-hand side is a multiple of 37 (since $q > 0$) and the left-hand side cannot be. This is a contradiction.

   (b) Suppose not: suppose $\sqrt[3]{2} = p/q$, for some coprime integers p and q. Then, cubing both sides, we get $2 = p^3/q^3$, and so $p^3 = 2q^3$. The right-hand side is a multiple of 2, and thus the left-hand side must be too, As a result, p is even: an odd number cubes to an odd number. So we can write $p = 2r$ for some r. Then we substitute in to get $(2r)^3 = 8r^3 = 2q^3$ which cancels to get $4r^3 = q^3$. The left-hand side is even, too, and hence the right-hand side must be too. Hence q is even, and so shares a factor (of 2) in common with p, which is a contradiction. Hence $\sqrt[3]{2}$ is irrational.

2. (a) Show directly from the definition of convergence that the sequence defined by

   $$a_n = \frac{1 - \frac{1}{n}}{1 + \frac{1}{n}}$$

   converges to 1.

   (b) Show directly from the definition of convergence that the sequence defined by

   $$b_n = \frac{(2n+1)(2n-1)}{n^2}$$

   converges to 4.

Solution

(a) We can simplify to get $a_n = \frac{n-1}{n+1} = 1 - \frac{2}{n+1}$. We aim to show that, for all ϵ, there is some N such that for all $n > N$ we have

$$|a_n - 1| < \epsilon,$$

but this rearranges to $\frac{2}{n+1} < \epsilon$. Hence if we take $N = \lceil 2/\epsilon - 1 \rceil$, then for $n > N$ we get $\frac{2}{n+1} < \frac{2}{2/\epsilon} = \epsilon$ as needed.

(b) The aim is to show that, for all ϵ, there is some N such that for all $n > N$ we have

$$|b_n - 4| < \epsilon.$$

But we have

$$\left|\frac{(2n+1)(2n-1)}{n^2} - 4\right| = \left|\frac{4n^2 - 1}{n^2} - 4\right|$$
$$= \left|4 - \frac{1}{n^2} - 4\right| = \left|-\frac{1}{n^2}\right| = \frac{1}{n^2}.$$

So the aim is to show that, for all ϵ, there is some N such that for all $n > N$ we have

$$\frac{1}{n^2} < \epsilon.$$

If we take $N = \lceil 1/\epsilon \rceil$, then for $n > N$ we have

$$\frac{1}{n^2} \leq \frac{1}{n} < \frac{1}{N} \leq \frac{1}{1/\epsilon} = \epsilon,$$

which is what we need.

3. **Challenge:** Show that $\sqrt{3} + \sqrt{5} + \sqrt{7}$ is irrational.

[*Please hand in attempts to the Challenge problem on a separate sheet of paper so they can make their way to Dr Cranch more easily.*]