

MAS114: Lecture 1

James Cranch

<http://cranch.staff.shef.ac.uk/mas114/>

2017–2018

Welcome!

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I'll be teaching it. Call me James if you want to be informal, or Dr Cranch if you feel a need to be formal.

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- ▶ Notes will be placed online on the course webpage several days before each lecture:

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- ▶ The course webpage also has some practical advice, including on what to do if you miss a lecture.

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If you do not do this, you *will not* be able to catch up in the run-up to the exams.

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- ▶ Leave it until the time of the exam.

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- ▶ *part* of mathematics,
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- ▶ *particularly useful* in applications outside mathematics.

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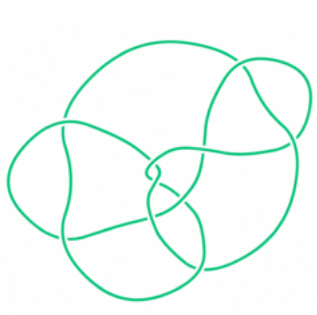
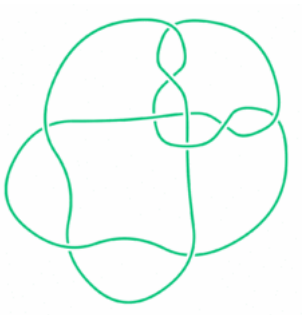
Ideas of space

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Here are two knotted loops of string:

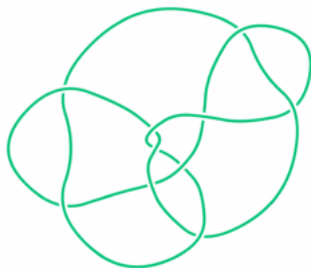
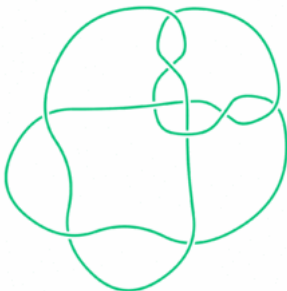
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Are they the same? That is, if I had one, could I manipulate it so as to look like the other?

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Ideas to do with space are nowadays of core importance in physics, just as numbers have. The world is made of space with interesting things in, after all.

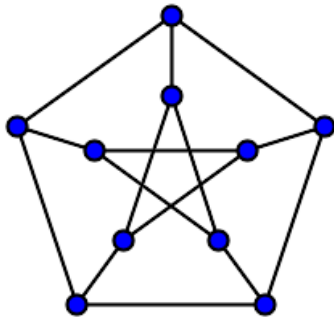
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Is it possible to have a party of ten people, where everyone is a friend or a friend-of-a-friend of everyone else, and where everyone has exactly three friends present?

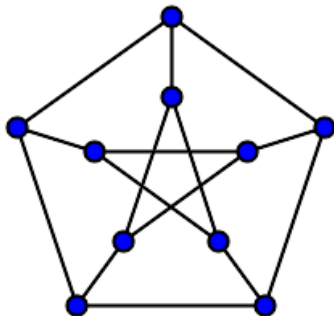
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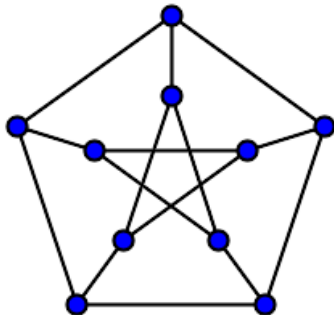
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It's true that the numbers three and ten appear in this problem. But it's not really a problem about numbers: it's a problem about social networks and how they can be configured.

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This has great application in computer science: after all, computer networks are examples of networks.

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Let's look at what that means.

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One has to be very careful, but the abstraction of mathematics has been an amazing tool. For example, it *may be* true that nothing is perfectly round, but many things are so nearly round as to make their real shape irrelevant.

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If we want to take liberties in our arguments then there's not much point in making an abstraction in the first place.

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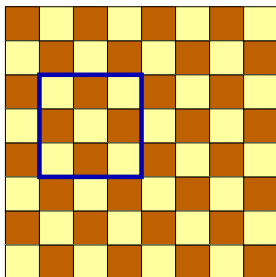
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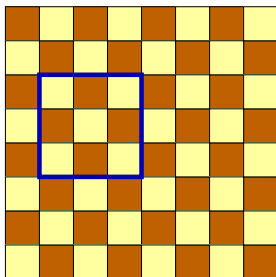
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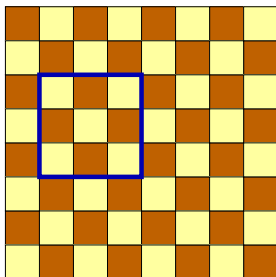


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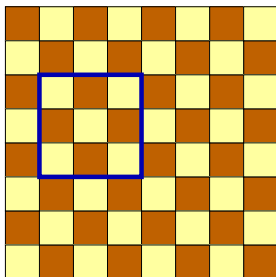


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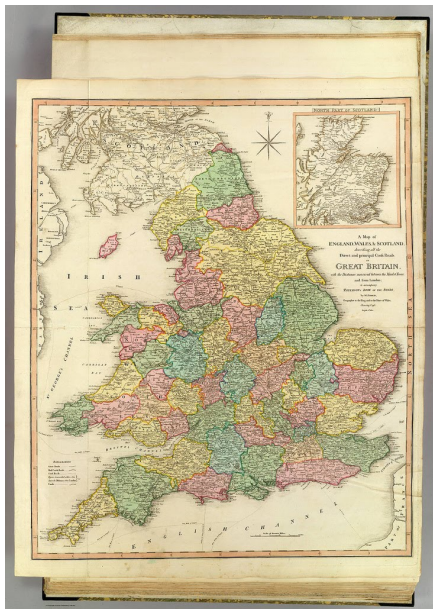
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Here's a vintage map of England and Wales (and, bizarrely, the Isle of Man) coloured in this way:



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- ▶ In **1890**, Percy Heawood pointed out that Kempe's proof contained a big mistake.
- ▶ In **1891**, Julius Petersen pointed out that Tait's proof also contained a big mistake. Now, after twelve years spent believing the problem had been solved, and the answer was yes, mathematicians realised that in fact, they still had no idea.

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- ▶ As of **2017**, the argument of Appel and Haken has been checked many times, and is accepted as a complete solution.

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At times this may seem like an unnecessary burden: especially when you feel that the right answer is “obvious”. However, if you don't spend time in shallow water learning how to swim, you'll never be comfortably able to swim in deep water.

Sets

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But in order to study them properly, we'll need to start at the beginning, by talking about *sets*.

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We often need to say which we mean, in order to avoid confusion and error. For example, it's certainly possible that I might invite 3 friends over for dinner, but it's hard to invite -5 friends or $3/4$ friends or $\sqrt{2}$ friends over.

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$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

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We'll see many more of those curly brackets later!