MAS114: Lecture 2

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2017–2018

Announcement 1: website

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- the course notes;
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- details of the online tests.

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Working with a bigger system of numbers can cure this.

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So the natural numbers are the same thing as the nonnegative integers.

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We write $\mathbb{Q}$ for the set of rational numbers (Q stands for "quotient", which is a name for what you get when you do division).
Of course, any integer n can be regarded as a rational (we can take $\frac{n}{1}$), so

$$\mathbb{Z} \subset \mathbb{Q}.$$

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There are still many things we might want to do but can't do in the rationals though: square roots, logarithms, trigonometry, and suchlike.

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The *real numbers* $\mathbb{R}$ are perhaps the most general sort of numbers you'll have used by now (or perhaps not). They contain lots of the numbers you care about, for example:

$$\pi \in \mathbb{R}, \qquad \log 1729 \in \mathbb{R}, \qquad \sqrt{5} \in \mathbb{R}, \qquad \sin(37^\circ) \in \mathbb{R}.$$

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Producing a good and useful definition of $\mathbb{R}$ is quite tricky, and there wasn't one until about 1870. We'll see one later in the course.

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- $a \in S$ to mean "a is in S".
- $a \notin S$ to mean "a is not in S".
- $|S|$ to denote the *size* of S: the number of elements in it. (Of course, some sets are infinite, but this works well for finite ones, at least.)

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Let's write some examples of facts about T using our notation:

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However, there are few good reasons to write something like that.

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That shouldn't confuse you. They're different for pretty much the same reason that "an empty bag" is not the same thing as "a bag which contains an empty bag and nothing else".

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Notice that, for every set A we have

$$A \subset A \qquad \text{and} \qquad \varnothing \subset A.$$