MAS114: Lecture 3

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2018–2019

Announcement: homework

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Have you found the homework online yet?

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You need to hand it in at the beginning of your problems class at the end of the week.

Unions

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Let A and B be sets. We define their *union* $A \cup B$ to contain exactly the things that are in one set or the other (or both):

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

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$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} .$$

That notation is called a *set comprehension*: the thing on the left of the vertical bar are the things we want to put in the set, and the things on the right of the vertical bar are the conditions under which we put them in. We'll use them a lot.

Intersections

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Similarly, we define the *intersection* $A \cap B$ to contain exactly the things that are in both sets:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Differences

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Lastly, we define the *difference* $A \backslash B$ to be the things which are in A but not in B:

$$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\} .$$

Equality of sets

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Proposition

Let A, B and C be three sets. We have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

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Proof.

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$\square$

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That was the first example of a formal proof in this course. You'll have to write many proofs like this yourself, in assessed homework and in the exam. Though we'll discuss it in depth later, it may be worth observing the style from the beginning. One big mistake that many beginner mathematicians make is *not using words to explain the flow of the argument*.

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Suppose there is a set S of all sets which are not elements of themselves:

$$S = \{A \mid A \notin A\}.$$

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However, you probably won't need to worry about this, unless you take a course in set theory later in your mathematical careers.

Functions

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Definition

Given sets A and B, a *function* (sometimes called a *map*) $f : A \to B$ gives for each element $a \in A$ a unique element $f(a) \in B$.

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- the function $g : \mathbb{Q} \to \{4, 6\}$ defined by

$$g(x) = \begin{cases} 4, & \text{if } x = 3/7 \text{ or } x = -14/17; \\ 6, & \text{otherwise.} \end{cases}$$

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- some people use it to mean the image;
- some (confused) people, who don't know the difference, use it inconsistently to mean both.