MAS114: Lecture 4

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Functions

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I was thinking about the function from the set of people in this room to the natural numbers.

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- they have the same domain and codomain, as $f, g : A \to B$; and
- their values are equal, for every point in the domain: in other words, for all $a \in A$, we have $f(a) = g(a)$.

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$$g \circ f(x) = g(f(x)).$$

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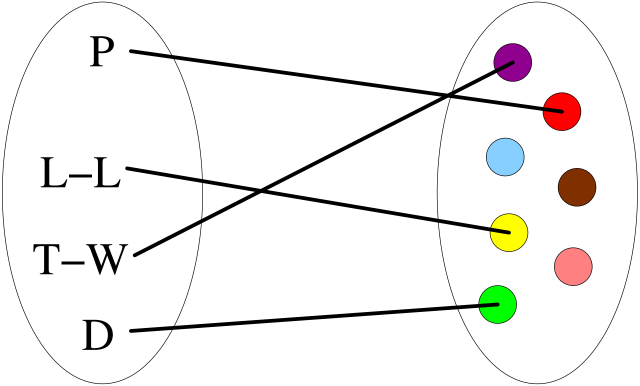
Accordingly, here is a function from the set T of Teletubbies, as considered earlier, to the set of colours:



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Here are some useful words.

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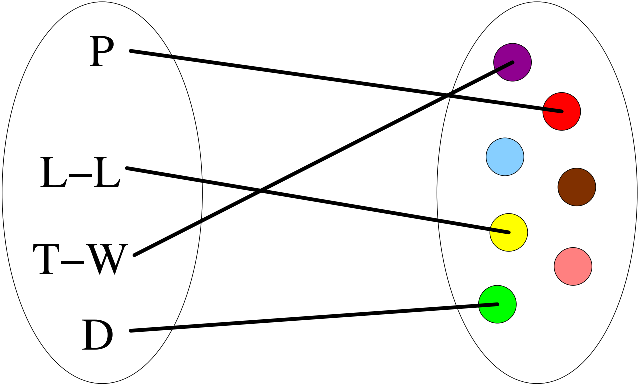

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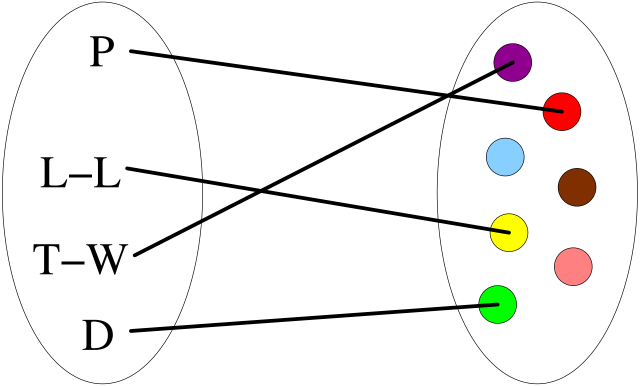

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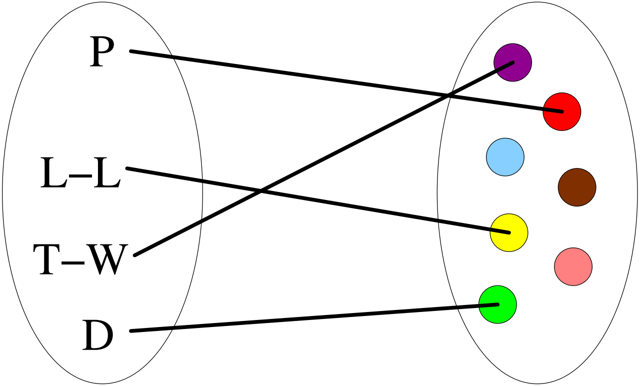

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However, it is not surjective, because there are no pink Teletubbies in all of Teletubbyland. Hence it is also not bijective.

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Note that these properties (injective, surjective, bijective) don't just depend on the rule that defines it: they depend on the domain and codomain.

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You'll see a lot more about inverses next semester.

Logic

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As such, it is quite different to saying "A is true, and therefore B is also true". Beginning students often get these confused.