

# MAS114: Lecture 4

James Cranch

<http://cranch.staff.shef.ac.uk/mas114/>

2017–2018

# Teaching Committee

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I was thinking about the function from the set of people in this room to the natural numbers.

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“unique element” A function must have only one value at any given element of the domain. If we set  $\beta(n)$  to be the real number  $x$  whose square is  $n$ , why does that not define a function  $\beta : \mathbb{N} \rightarrow \mathbb{R}$ ?

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- ▶ their values are equal, for every point in the domain: in other words, for all  $a \in A$ , we have  $f(a) = g(a)$ .

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$$g \circ f(x) = g(f(x)).$$

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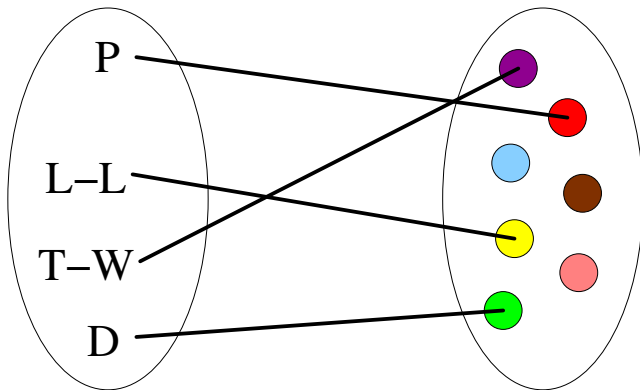
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Here are some useful words.

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## Definition

A function  $f : A \rightarrow B$  is said to be *injective* if, for any two elements  $a_1, a_2 \in A$  with  $a_1 \neq a_2$ , then  $f(a_1) \neq f(a_2)$ .

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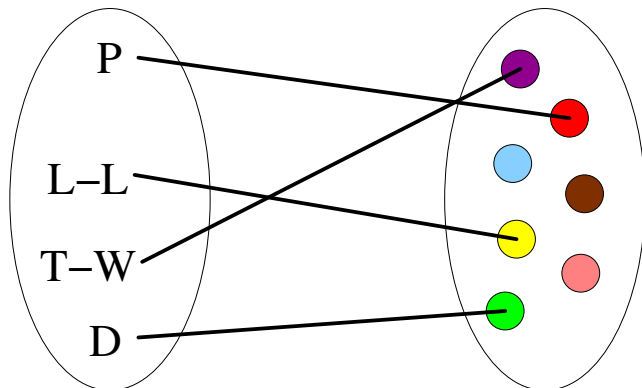
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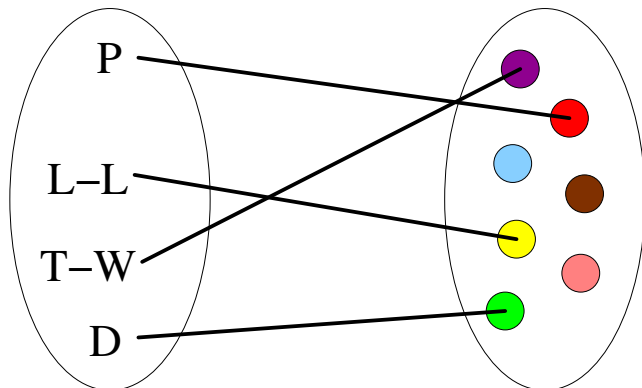
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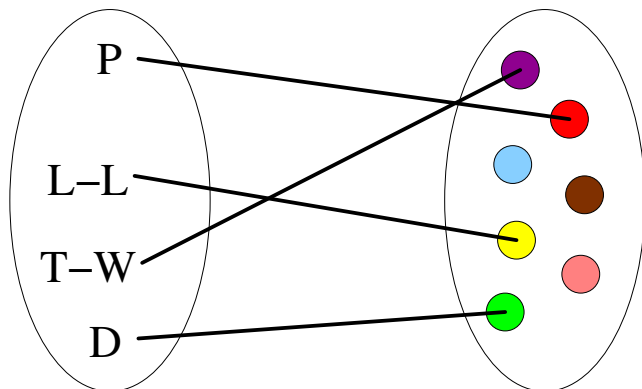


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It is injective, because each one of the Teletubbies has a different colour.

However, it is not surjective, because there are no pink Teletubbies in all of Teletubbyland. Hence it is also not bijective.



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Note that these properties (injective, surjective, bijective) don't just depend on the rule that defines it: they depend on the domain and codomain.

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You'll see a lot more about inverses next semester.

# Logic

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As such, it is quite different to saying “ $A$  is true, and therefore  $B$  is also true”. Beginning students often get these confused.