MAS114: Lecture 5

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2017–2018

Implication

We were discussing the concept of implication: writing $A \Rightarrow B$ to mean "if A, then B".

More caveats

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For example, it's correct to say

> *"If $2 + 2 = 337$, then this course is lectured by Dr Cranch".*

This may be a surprise if you're basing your intuition on ordinary English, where people use the words "if. . . then" in several different ways, sometimes slightly ambiguously.

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So it's correct to say

> *If it rains next Wednesday, then* $2 + 2 = 4$.

In fact, it's true that $2 + 2 = 4$ no matter whether it rains next Wednesday, but that's not a problem. Whether or not it's *helpful* to say that is another question!

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Definition

Consider a statement of the form $A \Rightarrow B$. Then the *converse* of that statement is the statement $B \Rightarrow A$.

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Sometimes people shorten "if and only if" to "iff".

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Note that double negation doesn't do anything: the statement $\neg(\neg P)$ is equivalent to P. Since statements are either true or false, if it's not "not true", it's true.

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Consider a statement of the form $P \Rightarrow Q$. Its *contrapositive* is the statement $(\neg Q) \Rightarrow (\neg P)$.

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For example,

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is probably intended to mean “but not both”. In mathematical argument when we use “or” and mean “but not both”, we have to say so explicitly.

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The only way that the first statement can be false is if P is true and Q is false. But that's also the only way that the second statement can be false, so they're equivalent. $\square$

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Note that that shows another style of proof of logical statements: by analysis rather than the "case bash" used in truth tables.

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$$\begin{aligned} \forall \quad & \text{for "for all";} \\ \exists \quad & \text{for "there exists";} \\ \mathrm{s.t.} \quad & \text{for "such that".} \end{aligned}$$

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is to be read as

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For example, the statement

$$\forall n \in \mathbb{N}, \quad \exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 = n$$

says that every natural number n has a square root x.