MAS114: Lecture 6

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$$\forall n \in \mathbb{N}, \quad \exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 = n$$

says that every natural number has a square root.

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The order of quantifiers is very important. If we swap over the two quantifiers in the last example, we get

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This says that there's a particular number x which has the property that x is the square root of *every natural number*. And that's nonsense.

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Similarly, the negation of “there exists a dolphin who likes Beethoven” is “there does not exist a dolphin who likes Beethoven”, and that’s equivalent to “all dolphins do not like Beethoven”.

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Perhaps you may want to remember that "negation swaps $\forall$ and $\exists$."

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Perhaps you may want to remember that "negation swaps $\forall$ and $\exists$." But being able to *do it correctly by remembering what's going on* is much more important than remembering a slogan. After a while it should come to seem natural.

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- *true*, in which case you need to prove it in general (that's a statement with a "$\forall$" in);
- *false*, in which case you need to find a counterexample (that's a statement with a "$\exists$" in).

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> *If I can reach the bottom (rung number zero?) of a ladder,*
> *and if I'm on any rung I can reach the next rung up,*
> *then I can reach any rung on the ladder.*

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Writing that out was okay, but you are probably glad I didn't write out a proof that we could reach the hundred and seventy-eighth rung. I suppose that we could do so, writing "and so on" at some point: but that's a little vague (what about situations where it isn't obvious what "and so on" means)?

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We call part (i) the *base case*, and part (ii) the *induction step*. These words agree quite well with our mental picture of a ladder! When we are trying to prove the induction step $P(k) \Rightarrow P(k+1)$ we refer to $P(k)$ as the *induction hypothesis*.

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We'll prove many things by induction in this course, but this is one:

Proposition

For any natural number n, we have the following formula for the sum of the first n positive integers:

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So $P(3)$ is the statement that says $1 + 2 + 3 = 3 \times 4/2$, and $P(10)$ is the statement that

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Notice that $P(n)$ is *not a number*, it's a *statement*.

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We will prove $P(n)$, which says that

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You may know other ways of proving that. (I can think of a few.) But I hope you're impressed with this as a strong potential method for proving identities.

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> If you believe that induction is a reliable method of proof (and I do, and I hope you do too), then it had better be the case that we're not using induction correctly.

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If you don't have a base case, such as $P(0)$, then it's of no use to prove that $P(k) \Rightarrow P(k+1)$ for all k. It's no use to be able to climb a ladder if the bottom of the ladder is unreachable.