

MAS114: Lecture 6

James Cranch

<http://cranch.staff.shef.ac.uk/mas114/>

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For example, the statement

$$\forall n \in \mathbb{N}, \exists x \in \mathbb{R} \text{ s.t. } x^2 = n$$

says that every natural number has a square root.

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This says that there's a particular number x which has the property that x is the square root of *every natural number*. And that's nonsense.

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Similarly, the negation of “there exists a dolphin who likes Beethoven” is “there does not exist a dolphin who likes Beethoven”, and that’s equivalent to “all dolphins do not like Beethoven”.

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Perhaps you may want to remember that “negation swaps \forall and \exists .” But being able to *do it correctly by remembering what’s going on* is much more important than remembering a slogan. After a while it should come to seem natural.

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In general, if you have a general statement and you don't know if whether it's true or false, then it could either be:

- ▶ *true*, in which case you need to prove it in general (that's a statement with a “ \forall ” in);
- ▶ *false*, in which case you need to find a counterexample (that's a statement with a “ \exists ” in).

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If I can reach the bottom (rung number zero?) of a ladder,

and if I'm on any rung I can reach the next rung up,
then I can reach any rung on the ladder.

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Writing that out was okay, but you are probably glad I didn't write out a proof that we could reach the hundred and seventy-eighth rung. I suppose that we could do so, writing "and so on" at some point: but that's a little vague (what about situations where it isn't obvious what "and so on" means)?

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When we are trying to prove the induction step $P(k) \Rightarrow P(k + 1)$ we refer to $P(k)$ as the *induction hypothesis*.

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We'll prove many things by induction in this course, but this is one:

Proposition

For any natural number n , we have the following formula for the sum of the first n positive integers:

$$1 + 2 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}.$$

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So $P(3)$ is the statement that says $1 + 2 + 3 = 3 \times 4/2$, and $P(10)$ is the statement that

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Notice that $P(n)$ is *not a number*, it's a *statement*.

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We will prove $P(n)$, which says that

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But I hope you're impressed with this as a strong potential method
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If you believe that induction is a reliable method of proof (and I do, and I hope you do too), then it had better be the case that we're not using induction correctly.

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If you don't have a base case, such as $P(0)$, then it's of no use to prove that $P(k) \Rightarrow P(k + 1)$ for all k . It's no use to be able to climb a ladder if the bottom of the ladder is unreachable.