MAS114: Lecture 7

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We'll take $P(1)$ as the base case of the induction. This is the statement "Given any one horse, all of them have the same colour": this is obviously true.

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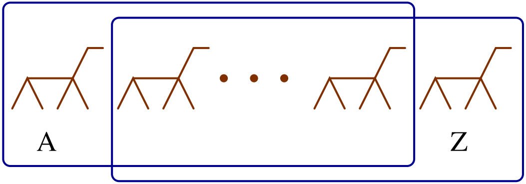

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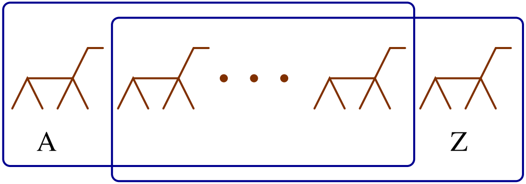

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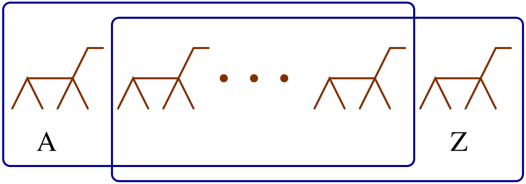

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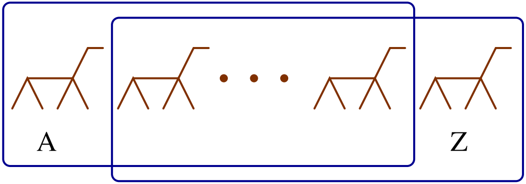

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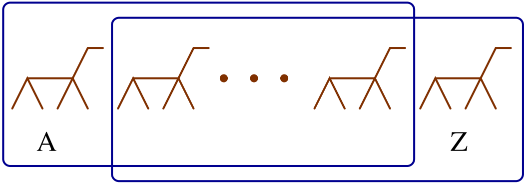

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In fact, it's a parody of a *valid* style of argument. If it is the case that *any two things are the same*, then we could prove using exactly this method that they're *all the same*. In fact, this is something you already know, since "all are alike" and "no two differ" are synonymous phrases.

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Perhaps you want to think of that as saying "if have a door which leads to the fifteenth rung of a ladder, and you know how to climb ladders, then you can get to every rung above the fifteenth".

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Actually, you should have been prepared for this variant: my induction proof that "all horses have the same colour" started with 1, not 0. (Okay, that proof was wrong. But there was nothing wrong with *that bit* of the proof: there's nothing wrong with induction starting from 1. It was something else that was wrong).

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In order to prove this, we assume $P(0), \ldots, P(k)$ are all true and have to prove that $P(0), \ldots, P(k+1)$ are all true. But then all of these except the last are assumptions: what is left is to prove $P(k+1)$ assuming $P(0), \ldots, P(k)$, and that's exactly the induction step of a strong induction.

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I like to think that the proof was arranged according to the shape of the definition of the Fibonacci numbers: that definition has two base cases $F_0 = 0$ and $F_1 = 1$, and a step $F_{n+2} = F_{n+1} + F_n$. This is not a rare coincidence.

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For example, consider the set of pairs (m, n) of naturals, where we say that $(m, n) < (m', n')$ if $m < m'$, or if $m = m'$ and $n < n'$.

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