

MAS114: Lecture 8

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A Horse of a Different Colour

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It is not too hard to prove that that is well-ordered: any set of pairs of natural numbers has a “least” element with respect to this ordering. Hence we can do (strong) induction on pairs of naturals!

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Also, *read it back* to yourself (or better yet, get a colleague to read it). You're writing it so others can read: it's good to test it to make sure this is possible. This goes particularly for proofs containing large amounts of symbols: these can be very hard to read, and reading it back to yourself is probably the best way of detecting this.

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Alternatives to symbols

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Here are some phrases which do the job of “:.”:

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Here are some phrases which do the job of “∴”:

so,

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When it's really important, I like to use both to make sure the point is clear. I might write,

Let $A = \sum_{i=1}^n a(i)$ be the sum of the first n values of a , and let $p > A$ be a prime number greater than A .

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Many novices write this, wanting it to mean:

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However, experienced readers will read it as

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This is of course nonsense: equations don't really have squares, and $5 = 25$ is an invalid equation anyway.

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$$\text{Let } P(n) = a_n = 3^n + 1.$$

This has $P(n)$ as a *number*, not a *statement*. Instead write:

$$\text{Let } P(n) \text{ be the statement that } a_n = 3^n + 1.$$

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It is particularly important to avoid unstated assumptions. For example, if a proof contains an assertion that some construction is a function, then the definition of a function gives you some things to check: that *every* element of the domain gives a *unique* element lying *inside* the codomain. Unless they're obviously true, it could be that these checks are the hardest and most interesting part of the proof. They could even be lies.

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Also, some explanation at the end is important. When you reach a conclusion, why do you think that what you have written actually means you have finished the proof?

If the proof is long, then regard it as being made of several parts. Give each part an explanation when you start and when you finish it.