MAS114: Lecture 12

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2017–2018

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The Fermat equation becomes more interesting because of our inability to reliably take nth roots in $\mathbb{Z}$ or $\mathbb{N}$: which x and y can we take for which this recipe works?

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This equation would be simple if we cared about real solutions: we could take any x we like and then just take $y = (120 - 39x)/54$. However, because we can't do division reliably in $\mathbb{Z}$, this recipe is not very helpful: how do we know which x will give us an integer y?

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or in other words, that $x = -440$, $y = 320$ gives a solution. Now, you might wonder whether this is the *only* solution.

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Common divisors

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Proposition

Let a and b be positive integers. Any common divisor of a and b is a divisor of the greatest common divisor.

Proof.

If $d \mid a$ and $d \mid b$, then $d \mid (a - qb)$ for any q. Hence d is a divisor of the numbers obtained after every step of Euclid's algorithm, and so it is a divisor of the gcd. $\square$

Common divisors

Here's a useful result about common divisors.

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Let a and b be positive integers. Any common divisor of a and b is a divisor of the greatest common divisor.

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All these things look pretty similar, and it's time we got ourselves a language for discussing these things better.

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We can now say that an even number is a number congruent to 0 (modulo 2), and an odd number is a number congruent to 1 (modulo 2).

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Arguments about time frequently involve understandings of congruences. For example, I was born on a Sunday, and the closing ceremony of the 2012 Summer Olympics took place on a Sunday too. So the number of days since the former is congruent to the number of days since the latter, modulo 7.

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Arguments about time frequently involve understandings of congruences. For example, I was born on a Sunday, and the closing ceremony of the 2012 Summer Olympics took place on a Sunday too. So the number of days since the former is congruent to the number of days since the latter, modulo 7.
Notice that saying that a is congruent to 0, modulo m, is exactly the same as saying that a is a multiple of m (since it's saying that $m \mid (a - 0)$).

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Proof (of some of them).

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