

MAS114: Lecture 19

James Cranch

<http://cranch.staff.shef.ac.uk/mas114/>

2017–2018

Maths tonight

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(As if you need any other reason to go), there'll be free pizza.

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But I want to flag that up as being possibly inappropriate: our aim in this section is to say something intelligent about systems of numbers bigger than \mathbb{Q} . We shouldn't even be confident that $\sqrt{2}$ exists yet.

However, thanks to this theorem, we can be confident at least that there's no number *inside* \mathbb{Q} which deserves to be called $\sqrt{2}$.

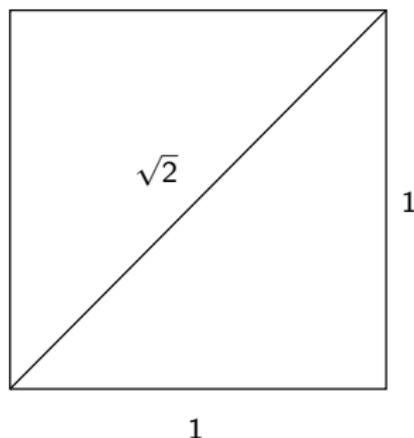
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This, to the Greeks, was evidence that there was a world beyond \mathbb{Q} ; a world of *irrational numbers* (numbers not in \mathbb{Q}). They needed a number called $\sqrt{2}$, so they could talk about the diagonal of a unit square:



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For the time being, and *for the time being only* we'll investigate the reals in a similar, informal way. For now, you can regard the real numbers \mathbb{R} as being built out of decimals (as you did at school). In the last lecture of the course, we'll sort this out, and consider a modern construction of the reals.

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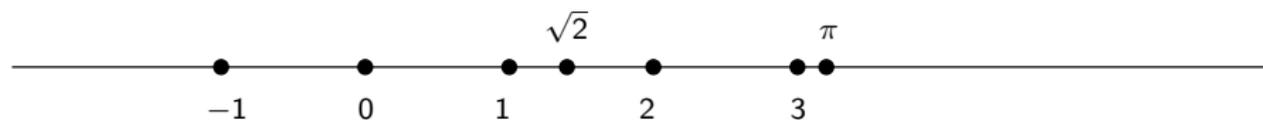
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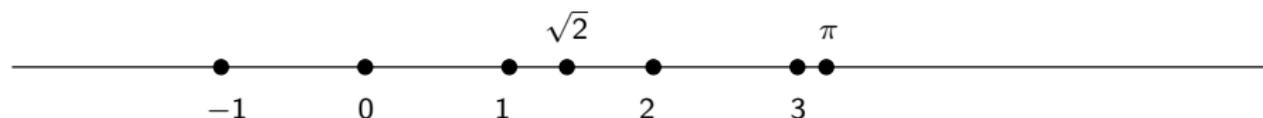
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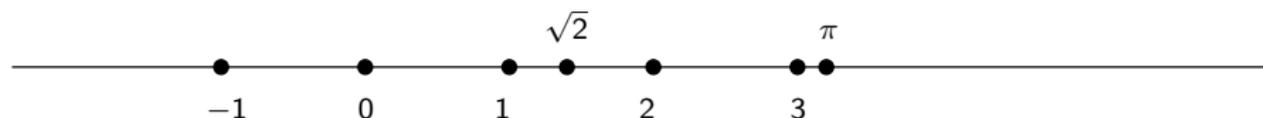
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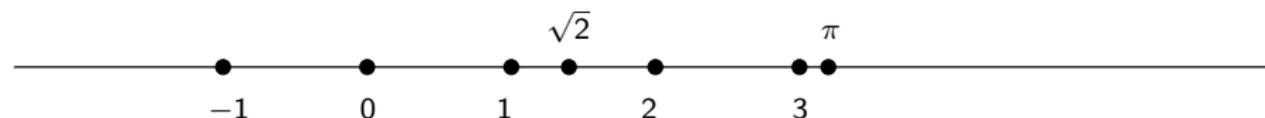


I've marked on the integers $-1, 0, 1, 2$ and 3 , which are all in \mathbb{Z} and hence in \mathbb{Q} .

I've also marked on $\sqrt{2}$, which we now know to be irrational, and π , which I've claimed to you is irrational: these things are in the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers.

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In my mind, I think of the real numbers \mathbb{R} as a solid line, and the rational numbers \mathbb{Q} as a very fine gauze net stretched out within it.

The rationals in \mathbb{R}

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The reals \mathbb{R} are also a lovely system of numbers, closed not just those four operations but many others: square roots (of positive numbers), sines, cosines, and so on.

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So, the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ really are just the big messy clump left over in \mathbb{R} when you remove \mathbb{Q} . It's a bit weird we even have a name for these: I don't know a good name for the set $\mathbb{Q} \setminus \mathbb{Z}$ of rationals which aren't integers.

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Proof.



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It turns out that the most interesting things you can ask about are to do with *approximation*. Why is the notion of approximation so important?

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The definition will seem complicated, and probably harder to get your head around than other definitions in the course. However, that's because it really is a subtle concept: all the simpler approaches you might think of are wrong.

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Of course, this sequence never gets particularly close to 1000 (the sequence never goes above 4, so it never gets within 996 of π).

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But this means that if our definition of “converging to x ” were the completely wrong definition “gets closer and closer to x ”, then the sequence

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would “converge to π ”, but it would also “converge to 1000”. But that’s not what we want: this sequence is a terrible way of getting to 1000, and an awesome way of getting to π .

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I like to think of it as an argument with a very dangerous and unpleasant *evil opponent*. The evil opponent gets to choose a (positive real) distance, and we win if the sequence gets within that distance of x , and we lose if it doesn't.

In order to be *sure* of winning, we have to know how to beat the evil opponent whatever they say.

Wrong approach 2, continued

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So, in investigating how close the sequence

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gets to 1000, then if the evil opponent is stupid enough to ask “does the sequence get within distance 100000?” we’ll win. But, being an evil *genius*, they probably won’t ask that. Instead they’ll ask “does the sequence get within distance 0.001 of 1000?”,

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In our minds, this says “no matter what ϵ our evil opponent chooses, we can find some term a_n of the sequence such that a_n is within ϵ of x ”.

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But according to the definition above it would “converge” to both, because it gets as close as you like to 1 and it also gets as close as you like to 2.