MAS114: Lecture 19

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2017–2018

Maths tonight

Maths tonight

Jonathan Jordan will be speaking about probability and social media tonight at 5pm in Hicks LT7.

Maths tonight

Jonathan Jordan will be speaking about probability and social media tonight at 5pm in Hicks LT7.
(As if you need any other reason to go), there'll be free pizza.

Comments

Comments

Remark

I felt obliged to word the statement of that theorem fairly carefully.

Comments

Remark

I felt obliged to word the statement of that theorem fairly carefully. What I wanted to say, of course, was:

The number $\sqrt{2}$ *is not in* $\mathbb{Q}$.

Comments

Remark

I felt obliged to word the statement of that theorem fairly carefully. What I wanted to say, of course, was:

The number $\sqrt{2}$ *is not in* $\mathbb{Q}$*.*

But I want to flag that up as being possibly inappropriate: our aim in this section is to say something intelligent about systems of numbers bigger than $\mathbb{Q}$.

Comments

Remark

I felt obliged to word the statement of that theorem fairly carefully. What I wanted to say, of course, was:

The number $\sqrt{2}$ is not in $\mathbb{Q}$.

But I want to flag that up as being possibly inappropriate: our aim in this section is to say something intelligent about systems of numbers bigger than $\mathbb{Q}$. We shouldn't even be confident that $\sqrt{2}$ exists yet.

Comments

Remark

I felt obliged to word the statement of that theorem fairly carefully. What I wanted to say, of course, was:

The number $\sqrt{2}$ *is not in* $\mathbb{Q}$*.*

But I want to flag that up as being possibly inappropriate: our aim in this section is to say something intelligent about systems of numbers bigger than $\mathbb{Q}$. We shouldn't even be confident that $\sqrt{2}$ exists yet.

However, thanks to this theorem, we can be confident at least that there's no number *inside* $\mathbb{Q}$ which deserves to be called $\sqrt{2}$.

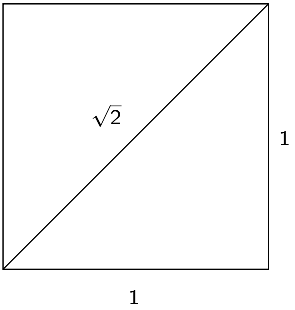
Irrational numbers

Irrational numbers

This, to the Greeks, was evidence that there was a world beyond $\mathbb{Q}$; a world of *irrational numbers* (numbers not in $\mathbb{Q}$).

Irrational numbers

This, to the Greeks, was evidence that there was a world beyond $\mathbb{Q}$; a world of *irrational numbers* (numbers not in $\mathbb{Q}$). They needed a number called $\sqrt{2}$, so they could talk about the diagonal of a unit square:



Irrational numbers today

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational.

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please...) that any one of the following numbers are irrational:

$$\pi + e,$$

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please. . .) that any one of the following numbers are irrational:

$$\pi + e, \quad \pi - e,$$

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please...) that any one of the following numbers are irrational:

$$\pi + e, \quad \pi - e, \quad \pi e,$$

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please. . .) that any one of the following numbers are irrational:

$$\pi + e, \quad \pi - e, \quad \pi e, \quad \pi/e,$$

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.
One high point includes the proof by Lambert in 1761 that π and e are irrational.
On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please...) that any one of the following numbers are irrational:

$$\pi + e, \quad \pi - e, \quad \pi e, \quad \pi / e, \quad \ln \pi,$$

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please...) that any one of the following numbers are irrational:

$$\pi + e, \quad \pi - e, \quad \pi e, \quad \pi / e, \quad \ln \pi, \quad e^e,$$

Irrational numbers today

Over the years, more and more examples were found of numbers which one might want to talk about, but which cannot be in $\mathbb{Q}$: various powers, logarithms, sines, cosines, and other constructions besides.

One high point includes the proof by Lambert in 1761 that π and e are irrational.

On the other hand, modern mathematics is still not particularly good, in general, at proving that numbers are irrational. For example, if you want to become famous, simply prove (please. . .) that any one of the following numbers are irrational:

$$\pi + e, \quad \pi - e, \quad \pi e, \quad \pi/e, \quad \ln \pi, \quad e^e, \quad e^{e^e}.$$

Investigating the reals

Investigating the reals

For centuries, the real numbers were considered in an informal way: nobody knew exactly how to define $\mathbb{R}$, but they knew what it ought to look like.

Investigating the reals

For centuries, the real numbers were considered in an informal way: nobody knew exactly how to define $\mathbb{R}$, but they knew what it ought to look like.
For the time being, and *for the time being only* we'll investigate the reals in a similar, informal way.

Investigating the reals

For centuries, the real numbers were considered in an informal way: nobody knew exactly how to define $\mathbb{R}$, but they knew what it ought to look like.

For the time being, and *for the time being only* we'll investigate the reals in a similar, informal way. For now, you can regard the real numbers $\mathbb{R}$ as being built out of decimals (as you did at school).

Investigating the reals

For centuries, the real numbers were considered in an informal way: nobody knew exactly how to define $\mathbb{R}$, but they knew what it ought to look like.

For the time being, and *for the time being only* we'll investigate the reals in a similar, informal way. For now, you can regard the real numbers $\mathbb{R}$ as being built out of decimals (as you did at school). In the last lecture of the course, we'll sort this out, and consider a modern construction of the reals.

A picture

A picture

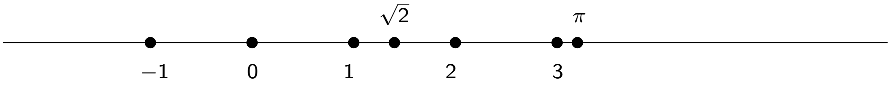
Our mental picture of the reals should be a picture of a numberline.

A picture

Our mental picture of the reals should be a picture of a numberline. Here's a numberline with some interesting points marked on:

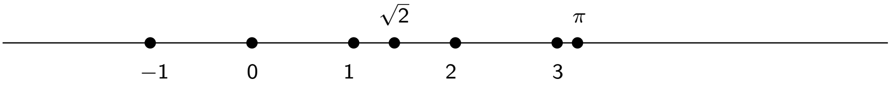
A picture

Our mental picture of the reals should be a picture of a numberline. Here's a numberline with some interesting points marked on:



A picture

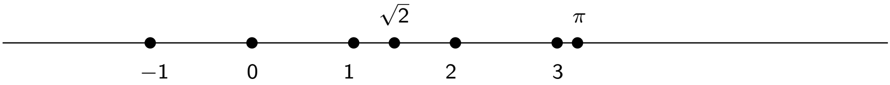
Our mental picture of the reals should be a picture of a numberline. Here's a numberline with some interesting points marked on:



I've marked on the integers -1, 0, 1, 2 and 3, which are all in $\mathbb{Z}$ and hence in $\mathbb{Q}$.

A picture

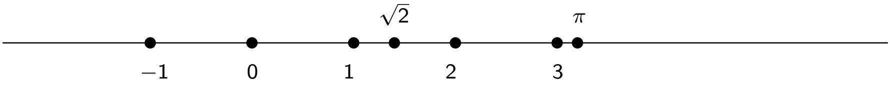
Our mental picture of the reals should be a picture of a numberline. Here's a numberline with some interesting points marked on:



I've marked on the integers -1, 0, 1, 2 and 3, which are all in $\mathbb{Z}$ and hence in $\mathbb{Q}$.
I've also marked on $\sqrt{2}$, which we now know to be irrational, and π, which I've claimed to you is irrational: these things are in the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers.

A picture

Our mental picture of the reals should be a picture of a numberline. Here's a numberline with some interesting points marked on:



I've marked on the integers -1, 0, 1, 2 and 3, which are all in $\mathbb{Z}$ and hence in $\mathbb{Q}$.
I've also marked on $\sqrt{2}$, which we now know to be irrational, and π, which I've claimed to you is irrational: these things are in the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers.
In my mind, I think of the real numbers $\mathbb{R}$ as a solid line, and the rational numbers $\mathbb{Q}$ as a very fine gauze net stretched out within it.

The rationals in $\mathbb{R}$

The rationals in $\mathbb{R}$

Bear in mind that that the rationals $\mathbb{Q}$ are a lovely system of numbers: we can add and subtract and multiply and divide rationals and remain inside the rationals.

The rationals in $\mathbb{R}$

Bear in mind that that the rationals $\mathbb{Q}$ are a lovely system of numbers: we can add and subtract and multiply and divide rationals and remain inside the rationals. Formally: if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $x + y$, $x - y$, xy and x/y (if y is nonzero) are all elements of $\mathbb{Q}$.

The rationals in $\mathbb{R}$

Bear in mind that that the rationals $\mathbb{Q}$ are a lovely system of numbers: we can add and subtract and multiply and divide rationals and remain inside the rationals. Formally: if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $x + y$, $x - y$, xy and x/y (if y is nonzero) are all elements of $\mathbb{Q}$. We say that $\mathbb{Q}$ is *closed* under addition, subtraction, multiplication and division.

The rationals in $\mathbb{R}$

Bear in mind that that the rationals $\mathbb{Q}$ are a lovely system of numbers: we can add and subtract and multiply and divide rationals and remain inside the rationals. Formally: if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $x + y$, $x - y$, xy and x/y (if y is nonzero) are all elements of $\mathbb{Q}$. We say that $\mathbb{Q}$ is *closed* under addition, subtraction, multiplication and division.
The reals $\mathbb{R}$ are also a lovely system of numbers, closed not just those four operations but many others: square roots (of positive numbers), sines, cosines, and so on.

The irrationals in $\mathbb{R}$

The irrationals in $\mathbb{R}$

The irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ are not a lovely system of numbers: they are not closed under any of these things.

The irrationals in $\mathbb{R}$

The irrational numbers $\mathbb{R} \backslash \mathbb{Q}$ are not a lovely system of numbers: they are not closed under any of these things.
For example, can we think of two irrational numbers whose sum is rational?

?

The irrationals in $\mathbb{R}$

The irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ are not a lovely system of numbers: they are not closed under any of these things.
For example, can we think of two irrational numbers whose sum is rational?

?

Can we think of two irrational numbers whose product is rational?

?

The irrationals in $\mathbb{R}$

The irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ are not a lovely system of numbers: they are not closed under any of these things.
For example, can we think of two irrational numbers whose sum is rational?

?

Can we think of two irrational numbers whose product is rational?

?

So, the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ really are just the big messy clump left over in $\mathbb{R}$ when you remove $\mathbb{Q}$.

The irrationals in $\mathbb{R}$

The irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ are not a lovely system of numbers: they are not closed under any of these things.
For example, can we think of two irrational numbers whose sum is rational?

?

Can we think of two irrational numbers whose product is rational?

?

So, the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ really are just the big messy clump left over in $\mathbb{R}$ when you remove $\mathbb{Q}$. It's a bit weird we even have a name for these: I don't know a good name for the set $\mathbb{Q}\backslash\mathbb{Z}$ of rationals which aren't integers.

Irrationals and rationals

Irrationals and rationals

However, at least the following is true (and almost obvious):

Irrationals and rationals

However, at least the following is true (and almost obvious):

Proposition

Let x be irrational, and y be rational. Then $x + y$ is irrational. Also, if y is nonzero, then xy is irrational.

Irrationals and rationals

However, at least the following is true (and almost obvious):

Proposition

Let x be irrational, and y be rational. Then $x + y$ is irrational. Also, if y is nonzero, then xy is irrational.

Proof.

?

□

Approximation

Approximation

Now our mission is to study the real numbers.

Approximation

Now our mission is to study the real numbers. When we were studying the integers, the main theme running through it all was to do with divisibility.

Approximation

Now our mission is to study the real numbers. When we were studying the integers, the main theme running through it all was to do with divisibility. Divisibility is, of course, not a very sensible thing to ask about over the reals.

Approximation

Now our mission is to study the real numbers. When we were studying the integers, the main theme running through it all was to do with divisibility. Divisibility is, of course, not a very sensible thing to ask about over the reals.

As a result, real analysis (the study of $\mathbb{R}$), and the questions which are interesting and helpful to ask, is very different to number theory.

Approximation

Now our mission is to study the real numbers. When we were studying the integers, the main theme running through it all was to do with divisibility. Divisibility is, of course, not a very sensible thing to ask about over the reals.

As a result, real analysis (the study of $\mathbb{R}$), and the questions which are interesting and helpful to ask, is very different to number theory.

It turns out that the most interesting things you can ask about are to do with *approximation*.

Approximation

Now our mission is to study the real numbers. When we were studying the integers, the main theme running through it all was to do with divisibility. Divisibility is, of course, not a very sensible thing to ask about over the reals.

As a result, real analysis (the study of $\mathbb{R}$), and the questions which are interesting and helpful to ask, is very different to number theory.

It turns out that the most interesting things you can ask about are to do with *approximation*. Why is the notion of approximation so important?

Convergence

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33 \cdots ,$$

the point is that the digits give a kind of address telling you how to find π on the numberline.

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33 \cdots ,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3,

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33\cdots ,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1,

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33\cdots ,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14,

Convergence

When we write that

$$\pi = 3.14159265358979323846264\,33\cdots ,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14, even closer still to 3.141, and so on.

Convergence

When we write that

$$\pi = 3.14159265358979323846264\,33\cdots,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14, even closer still to 3.141, and so on. The notion of *convergence*, which I'll define shortly, is a way of encoding this concept of increasingly good approximation.

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33\cdots,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14, even closer still to 3.141, and so on. The notion of *convergence*, which I'll define shortly, is a way of encoding this concept of increasingly good approximation. We will say that the series of rational numbers

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

"converges to π".

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33\cdots,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14, even closer still to 3.141, and so on.
The notion of *convergence*, which I'll define shortly, is a way of encoding this concept of increasingly good approximation. We will say that the series of rational numbers

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

"converges to π". That's supposed to mean that if you follow the address, you'll end up homing in on π.

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33\cdots ,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14, even closer still to 3.141, and so on.
The notion of *convergence*, which I'll define shortly, is a way of encoding this concept of increasingly good approximation. We will say that the series of rational numbers

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

"converges to π". That's supposed to mean that if you follow the address, you'll end up homing in on π.
The definition will seem complicated, and probably harder to get your head around than other definitions in the course.

Convergence

When we write that

$$\pi = 3.14159265358979323846264 33\cdots,$$

the point is that the digits give a kind of address telling you how to find π on the numberline. The number π is close to 3, closer to 3.1, closer still to 3.14, even closer still to 3.141, and so on.
The notion of *convergence*, which I'll define shortly, is a way of encoding this concept of increasingly good approximation. We will say that the series of rational numbers

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

"converges to π". That's supposed to mean that if you follow the address, you'll end up homing in on π.
The definition will seem complicated, and probably harder to get your head around than other definitions in the course. However, that's because it really is a subtle concept: all the simpler approaches you might think of are wrong.

Wrong approach 1

Wrong approach 1

The most obvious wrong definition is this:

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \dots$$

also gets closer and closer to 1000:

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

also gets closer and closer to 1000:

$$1000 - 3 \qquad = 997$$

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

also gets closer and closer to 1000:

$$\begin{aligned} 1000 - 3 &= 997 \\ 1000 - 3.1 &= 996.9 \end{aligned}$$

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

also gets closer and closer to 1000:

$$\begin{aligned} 1000 - 3 &= 997 \\ 1000 - 3.1 &= 996.9 \\ 1000 - 3.14 &= 996.86 \end{aligned}$$

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

also gets closer and closer to 1000:

$$\begin{aligned} 1000 - 3 &= 997 \\ 1000 - 3.1 &= 996.9 \\ 1000 - 3.14 &= 996.86 \\ 1000 - 3.141 &= 996.859 \end{aligned}$$

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

also gets closer and closer to 1000:

$$\begin{aligned} 1000 - 3 &= 997 \\ 1000 - 3.1 &= 996.9 \\ 1000 - 3.14 &= 996.86 \\ 1000 - 3.141 &= 996.859 \\ 1000 - 3.1415 &= 996.8585 \end{aligned}$$

Wrong approach 1

The most obvious wrong definition is this:

Completely wrong definition

A sequence *converges to* x if it gets closer and closer to x.

Why is this completely wrong? Well, for example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

also gets closer and closer to 1000:

$$\begin{aligned} 1000 - 3 &= 997 \\ 1000 - 3.1 &= 996.9 \\ 1000 - 3.14 &= 996.86 \\ 1000 - 3.141 &= 996.859 \\ 1000 - 3.1415 &= 996.8585 \end{aligned}$$

Of course, this sequence never gets particularly close to 1000 (the sequence never goes above 4, so it never gets within 996 of π).

Wrong approach 1, continued

Wrong approach 1, continued

But this means that if our definition of "converging to x" were the completely wrong definition "gets closer and closer to x", then the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

would "converge to π",

Wrong approach 1, continued

But this means that if our definition of “converging to x” were the completely wrong definition “gets closer and closer to x”, then the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

would “converge to π”, but it would also “converge to 1000”.

Wrong approach 1, continued

But this means that if our definition of “converging to x” were the completely wrong definition “gets closer and closer to x”, then the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

would “converge to π”, but it would also “converge to 1000”. But that’s not what we want: this sequence is a terrible way of getting to 1000, and an awesome way of getting to π.

Wrong approach 2

Wrong approach 2

Here's a slightly better idea:

Wrong approach 2

Here's a slightly better idea:

Completely wrong definition

A sequence *converges to* x if it gets as close as you like to x.

Wrong approach 2

Here's a slightly better idea:

Completely wrong definition

A sequence *converges to* x if it gets as close as you like to x.

Before we understand why this is wrong, we ought to make sure we know what this is supposed to mean.

Wrong approach 2

Here's a slightly better idea:

Completely wrong definition

A sequence *converges to* x if it gets as close as you like to x.

Before we understand why this is wrong, we ought to make sure we know what this is supposed to mean.
I like to think of it as an argument with a very dangerous and unpleasant *evil opponent*.

Wrong approach 2

Here's a slightly better idea:

Completely wrong definition

A sequence *converges to* x if it gets as close as you like to x.

Before we understand why this is wrong, we ought to make sure we know what this is supposed to mean.
I like to think of it as an argument with a very dangerous and unpleasant *evil opponent*. The evil opponent gets to choose a (positive real) distance, and we win if the sequence gets within that distance of x, and we lose if it doesn't.

Wrong approach 2

Here's a slightly better idea:

Completely wrong definition

A sequence *converges to* x if it gets as close as you like to x.

Before we understand why this is wrong, we ought to make sure we know what this is supposed to mean.
I like to think of it as an argument with a very dangerous and unpleasant *evil opponent*. The evil opponent gets to choose a (positive real) distance, and we win if the sequence gets within that distance of x, and we lose if it doesn't.
In order to be *sure* of winning, we have to know how to beat the evil opponent whatever they say.

Wrong approach 2, continued

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win.

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that.

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?",

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \dots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say.

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \dots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?",

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \dots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win.

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \dots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π,

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win.

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \dots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win. If they ask a question with an even tinier positive number in,

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.

On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win. If they ask a question with an even tinier positive number in, we just take more digits and use that and say "yes".

Wrong approach 2, continued

So, in investigating how close the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win. But, being an evil *genius*, they probably won't ask that. Instead they'll ask "does the sequence get within distance 0.001 of 1000?", and we'll lose, because it never gets anywhere near there.
On the other hand, if we're investigating how close the sequence gets to π, we win no matter what they say. If they ask "does the sequence get within distance 1000000 of π?", then we say "yes, 3 is within 1000000 of π", and win. If they ask "does the sequence get within distance 0.0001 of π, then we say "yes, 3.14159 is within 0.0001 of π", and win. If they ask a question with an even tinier positive number in, we just take more digits and use that and say "yes". We're happy: the evil opponent will not beat us.

Wrong approach 2, continued further

Wrong approach 2, continued further

Formally, if we have a sequence

$$a_0, \quad a_1, \quad a_2, \quad a_3, \ldots$$

that gets very close to x in this sense, one could specify this concept as

Wrong approach 2, continued further

Formally, if we have a sequence

$$a_0, \quad a_1, \quad a_2, \quad a_3, \ldots$$

that gets very close to x in this sense, one could specify this concept as

$$\forall \epsilon > 0, \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad |a_n - x| < \epsilon.$$

Wrong approach 2, continued further

Formally, if we have a sequence

$$a_0, \quad a_1, \quad a_2, \quad a_3, \ldots$$

that gets very close to x in this sense, one could specify this concept as

$$\forall \epsilon > 0, \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad |a_n - x| < \epsilon.$$

In our minds, this says "no matter what ϵ our evil opponent chooses, we can find some term a_n of the sequence such that a_n is within ϵ of x".

Why it's wrong

Why it's wrong

This is a much better concept. But it's still not right.

Why it's wrong

This is a much better concept. But it's still not right.
Here's an example of what could go wrong. Consider the sequence

$$1.1, \quad 2.01, \quad 1.001, \quad 2.0001, \quad 1.00001, \quad 2.000001, \quad \ldots$$

Why it's wrong

This is a much better concept. But it's still not right.
Here's an example of what could go wrong. Consider the sequence

$$1.1, \quad 2.01, \quad 1.001, \quad 2.0001, \quad 1.00001, \quad 2.000001, \quad \ldots$$

Does it converge to 1? Does it converge to 2?

Why it's wrong

This is a much better concept. But it's still not right.
Here's an example of what could go wrong. Consider the sequence

$$1.1, \quad 2.01, \quad 1.001, \quad 2.0001, \quad 1.00001, \quad 2.000001, \quad \ldots$$

Does it converge to 1? Does it converge to 2?
It's lousy as an address for either: if we give these instructions to the numberline's village postal worker, they'll get very annoyed as they keep walking from somewhere near 1 to somewhere near 2 and back again.

Why it's wrong

This is a much better concept. But it's still not right.
Here's an example of what could go wrong. Consider the sequence

$$1.1, \quad 2.01, \quad 1.001, \quad 2.0001, \quad 1.00001, \quad 2.000001, \quad \ldots$$

Does it converge to 1? Does it converge to 2?
It's lousy as an address for either: if we give these instructions to the numberline's village postal worker, they'll get very annoyed as they keep walking from somewhere near 1 to somewhere near 2 and back again.
But according to the definition above it would "converge" to both, because it gets as close as you like to 1 and it also gets as close as you like to 2.