MAS114: Lecture 20

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So that says "no matter what positive real ϵ our evil opponent gives us, we can point out some N, such that all the terms $a_{N+1}, a_{N+2}, a_{N+3}, \ldots$ are all within ϵ of x".

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That does an excellent job of making precise the concept of "gets close and stays close forever", and it's the right definition!

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Does that converge to 1000? No, it never comes within 1 of 1000 (for example), so it certainly doesn't stay within 1 of 1000 forever.

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$$a_0 = 1.1, \quad a_1 = 2.01, \quad a_2 = 1.001, \quad a_3 = 2.0001, \quad \ldots?$$

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Similarly, it doesn't converge to 2, because while it's sometimes close to 2, it's sometimes close to 1. So there is no N where a_n is always within 0.1 of 2 for all $n > N$: all the even-numbered a_n aren't in that range.

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So, given the difficulties we've had in finding the right definition, perhaps you'll have some sympathy for the fact that it took about two centuries to sort real analysis out properly. In what remains of the course I'll try to make you like this definition.

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This embodies the following slogan:

> *The distance from x to z if we go direct is less than if we go via y.*

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Let's now try proving that some sequence or other does converge, as we're not well practiced at that yet:

Proposition

The sequence

$$0, \quad 1/2, \quad 2/3, \quad 3/4, \quad 4/5, \quad \ldots$$

where $a_n = \frac{n-1}{n}$*, converges to* 1*.*

Rough version.

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exactly as required. $\square$

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That second version is obviously correct, and all the reasoning goes in the right direction. But analysis proofs often have the property that the best proof seems a bit mysterious. It's best to do the rough work and then rewrite it neatly.

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This is a course about fundamental techniques in mathematics and their proofs: if I set problems about convergence in MAS114, I need you to give a rigorous proof, only the definition of convergence (unless you're told otherwise), rather than using the slightly vaguer methods and extra theorems you saw there!