MAS114: Lecture 21

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Sad news

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There are no more online tests this semester.

Another convergence example

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Proposition

The sequence defined by $a_n = \frac{3^{n+1}}{3^n+1}$ *converges to* 3.

Rough version.

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One useful one is the following:

Theorem (Sandwich Lemma)

Suppose we have three sequences: $a_0, a_1, a_2, \ldots$*, and* $b_0, b_1, b_2, \ldots$ *and* $c_0, c_1, c_2, \ldots$*, such that:*

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Then the sequence $(b_i)_{i \in \mathbb{N}}$ also converges to x.

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However, we can proceed by sandwiching it between $a_n = 3 - \frac{1}{n}$ and $c_n = 3 + \frac{1}{n}$ (since all values of sin are always between -1 and 1). Showing that those two sequences both converge to 3 is simple (it's much like proofs we've done already).

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In vaguer terms, a sequence is Cauchy if no matter what we mean by close, there is some point beyond which all the terms of the sequence are close to each other.

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Of course, usually we have to choose N in a way which actually depends on ϵ; it's only in very special cases like these that we can choose one N for every ϵ.