MAS114: Lecture 22

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Proof.

$\boxed{?}$

$\square$

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Let's give it in context. What follows is *revisionist history*: things didn't actually happen exactly like this, but maybe they should have done.

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$$a-b=c-d \qquad \text{if and only if } a+d=b+c.$$

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  Again, we don't actually want one new number for each Cauchy sequence. There are other Cauchy sequences of rationals that converge to π (some of them more interesting, perhaps). A famous example is due to Gregory and Leibniz:

$$4, \quad 4-\frac{4}{3}, \quad 4-\frac{4}{3}+\frac{4}{5}, \quad 4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}, \quad \ldots.$$

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We demand that something be true for "every positive ϵ". That had better mean "every positive *rational* ϵ". In fact it won't make any difference in meaning.
But then, given that, we could just say that the reals *are* the Cauchy sequences of rationals, subject to some restriction about which ones are the same.

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We might regard this as saying "no matter what is meant by close, the two sequences get close to each other and stay close to each other forever".

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In particular, I hope you agree that it's a more natural way of understanding the reals than talking about decimal expansions.

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(This happens to be the decimal expansion of the square root of 2.) Then this can be accommodated in our construction easily, using the trick I mentioned earlier: we can represent it as the limit of the Cauchy sequence

$$1, \quad \frac{14}{10}, \quad \frac{141}{100}, \quad \frac{1414}{1000}, \quad \frac{14142}{10000}, \quad \ldots.$$

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If we start with 1 as an approximation, then this gives us the sequence

$$1, \quad \frac{3}{2}, \quad \frac{19}{12}, \quad \frac{577}{408}, \quad \frac{665857}{470832}, \ldots$$

It's not hard to imagine that this is a much better way of describing $\sqrt{2}$ than its decimal expansion: easier to prove things about it than some weird string of digits.

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Sooner or later, you can try using the same techniques in more exotic surroundings: the concepts of approximation we started talking about in this course give us a way in to studying abstract concepts of space.