

MAS114: Lecture 22

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Proof.



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Let's give it in context. What follows is *revisionist history*: things didn't actually happen exactly like this, but maybe they should have done.

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$$a - b = c - d \quad \text{if and only if} \quad a + d = b + c.$$

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So, for example, we get π as the limit of the Cauchy sequence of rationals

$$\begin{array}{cccccc} 3, & 3.1, & 3.14, & 3.141, & 3.1415, & \dots \\ 3, & \frac{31}{10}, & \frac{314}{100}, & \frac{3141}{1000}, & \frac{31415}{10000}, & \dots \end{array}$$

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Again, we don't actually want one new number for each Cauchy sequence. There are other Cauchy sequences of rationals that converge to π (some of them more interesting, perhaps). A famous example is due to Gregory and Leibniz:

$$4, \quad 4 - \frac{4}{3}, \quad 4 - \frac{4}{3} + \frac{4}{5}, \quad 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}, \quad \dots$$

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But then, given that, we could just say that the reals *are* the Cauchy sequences of rationals, subject to some restriction about which ones are the same.

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We might regard this as saying “no matter what is meant by close, the two sequences get close to each other and stay close to each other forever”.

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(This happens to be the decimal expansion of the square root of 2.) Then this can be accommodated in our construction easily, using the trick I mentioned earlier: we can represent it as the limit of the Cauchy sequence

$$1, \quad \frac{14}{10}, \quad \frac{141}{100}, \quad \frac{1414}{1000}, \quad \frac{14142}{10000}, \quad \dots$$

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If we start with 1 as an approximation, then this gives us the sequence

$$1, \quad \frac{3}{2}, \quad \frac{19}{12}, \quad \frac{577}{408}, \quad \frac{665857}{470832}, \dots$$

It's not hard to imagine that this is a much better way of describing $\sqrt{2}$ than its decimal expansion: easier to prove things about it than some weird string of digits.

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So far as number theory goes, our elementary methods will give out sooner or later. A good next step is to learn lots of algebra. (There are other good reasons to do that.) Next semester, Sam will take this up. You can return later in your degrees to a huge range of questions of which equations are solvable in which systems of numbers.

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Integration is a bit more technical, and will require more work. Sooner or later, you can try using the same techniques in more exotic surroundings: the concepts of approximation we started talking about in this course give us a way in to studying abstract concepts of space.