MAS114: Lecture 2

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

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Online tests

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$$\mathbb{N} = \{0, 1, 2, 3, \dots\} .$$

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We'll see many more of those curly brackets later!

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Working with a bigger system of numbers can cure this.

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We often use the handy words *non-negative*, meaning “not negative” (in other words, positive or zero) and *non-positive*, meaning “not positive” (in other words, negative or zero).
So the natural numbers are the same thing as the nonnegative integers.

The integers, continued

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Of course, any integer n can be regarded as a rational (we can take $\frac{n}{1}$), so

$$\mathbb{Z} \subset \mathbb{Q}.$$

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There are still many things we might want to do but can't do in the rationals though: square roots, logarithms, trigonometry, and suchlike.

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The *real numbers* $\mathbb{R}$ are perhaps the most general sort of numbers you'll have used by now (or perhaps not). They contain lots of the numbers you care about, for example:

$$\pi \in \mathbb{R}, \qquad \log 1729 \in \mathbb{R}, \qquad \sqrt{5} \in \mathbb{R}, \qquad \sin(37^\circ) \in \mathbb{R}.$$

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Producing a good and useful definition of $\mathbb{R}$ is quite tricky, and there wasn't one until about 1870. We'll see one later in the course.

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- $a \in S$ to mean "a is in S".
- $a \notin S$ to mean "a is not in S".
- $|S|$ to denote the *size* of S: the number of elements in it. (Of course, some sets are infinite, but this works well for finite ones, at least.)

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Let's write some examples of facts about T using our notation:

$$\textit{Po} \in T, \qquad \textit{Noo-noo} \notin T, \qquad |T| = 4$$

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However, there are few good reasons to write something like that.

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Note that $\varnothing$ is very different to $\{\varnothing\}$. The former, as I mentioned, has no elements; the latter has exactly one element.
That shouldn't confuse you. They're different for pretty much the same reason that "an empty bag" is not the same thing as "a bag which contains an empty bag and nothing else".

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Notice that, for every set A we have

$$A \subset A \qquad \text{and} \qquad \varnothing \subset A.$$

Unions

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That notation is called a *set comprehension*: the thing on the left of the vertical bar are the things we want to put in the set, and the things on the right of the vertical bar are the conditions under which we put them in. We'll use them a lot.

Intersections

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Similarly, we define the *intersection* $A \cap B$ to contain exactly the things that are in both sets:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} .$$

Differences

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Lastly, we define the *difference* $A \backslash B$ to be the things which are in A but not in B:

$$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\} .$$