MAS114: Lecture 3

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2021–2022

Announcement: homework

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I'd greatly prefer it if you wrote it by hand (and submitted a scan, if you're handing it in online). If that's a problem, let me know.

Equality of sets

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A straightforward way of proving this is often to show that $A \subset B$ and $B \subset A$. That is, in words, two sets A and B are equal if every element of A is an element of B and every element of B is an element of A.

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Proposition

Let A, B and C be three sets. We have

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Proof.

We'll show that each side is contained in the other: first we'll prove that

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C),$$

and then that

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C).$$

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$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof.

For the first one, suppose that $x \in A \cap (B \cup C)$; we must prove that $x \in (A \cap B) \cup (A \cup C)$.

Since $x \in A \cap (B \cup C)$, we have both $x \in A$ and $x \in B \cup C$, which in turn means that $x \in B$ or $x \in C$. In either case, the desired result holds:

- If $x \in B$, then since $x \in A$ also, then $x \in A \cap B$, and so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in C$, then, similarly, since $x \in A$, we have $x \in A \cap C$, and hence $x \in (A \cap B) \cup (A \cap C)$.

So we've proved that containment.

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Proposition

Let A, B and C be three sets. We have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof.

Now let's prove the other containment: that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. We suppose that $x \in (A \cap B) \cup (A \cap C)$ and must prove that $x \in A \cap (B \cup C)$. Since $x \in (A \cap B) \cup (A \cap C)$, we have either $x \in A \cap B$ or $x \in A \cap C$. In either case, we get what we want.

- If $x \in A \cap B$, then $x \in A$ and $x \in B$. From the latter, we get that $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.
- If $x \in A \cap C$, then, as before, $x \in A$ but now $x \in C$. But we still have $x \in B \cup C$, and so $x \in A \cap (B \cup C)$. □

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That was the first example of a formal proof in this course. You'll have to write many proofs like this yourself, in assessed homework and in the exam. Though we'll discuss it in depth later, it may be worth observing the style from the beginning. One big mistake that many beginner mathematicians make is *not using words to explain the flow of the argument*.

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Paradox

Suppose there is a set S of all sets which are not elements of themselves:

$$S = \{A \mid A \notin A\} .$$

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This creates a contradiction.

Proof.

Is S a member of itself? If $S \in S$, then by the definition of S, we have $S \notin S$. On the other hand, if $S \notin S$, then again by the definition of S we have $S \in S$.

$\square$

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However, you probably won't need to worry about this, unless you take a course in set theory later in your mathematical careers.

Functions

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Definition

Given sets A and B, a *function* (sometimes called a *map*) $f : A \to B$ gives for each element $a \in A$ a unique element $f(a) \in B$.

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- the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 - 7$.
- the function $g : \mathbb{Q} \to \{4, 6\}$ defined by

$$g(x) = \begin{cases} 4, & \text{if } x = 3/7 \text{ or } x = -14/17; \\ 6, & \text{otherwise.} \end{cases}$$

Domain and codomain

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- some people use it to mean the codomain;
- some people use it to mean the image;
- some (confused) people, who don't know the difference, use it inconsistently to mean both.

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$\gamma(4) = -3$ does not lie inside $\mathbb{N}$.

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- they have the same domain and codomain, as $f, g : A \to B$; and
- their values are equal, for every point in the domain: in other words, for all $a \in A$, we have $f(a) = g(a)$.